LIMIT ANALYSIS OF REINFORCED CONCRETE CYLINDRICAL SHELL ROOFS

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By
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CERTIFICATE

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NOTATION

The following symbols are used in this thesis. The list contains all the more important of the symbols that appear in the text.

A_{st} = Area of steel per unit length

$\bar{a}$ = Factor defining the position of the total longitudinal force developed in the edge beam from its top

$[A]$ = Rectangular matrix having coefficients of the equilibrium equations in the finite difference form as its elements

B = Chord width of a cylindrical shell

b = Constant in the straight line equation of the linearized M_ϕ-N_ϕ yield criterion

$[B]$ = Square matrix containing the elements corresponding to the independent columns of the matrix $[A]$

$[B]^{-1}$ = Inverse of the matrix $[B]$

c = Slope of the straight line equation of the linearized M_ϕ-N_ϕ yield criterion

c_1, c_2, c_3 = Constants defining the yielding of a material

$[C]$ = Rectangular matrix containing the elements corresponding to the dependent columns of the matrix $[A]$

$\{D\}$ = Column vector containing the right hand side values of the equilibrium equations in finite difference form as its elements

F_x = Longitudinal force developed in the edge beam at any section x

F_o = Longitudinal force developed in the edge beam at its centre

$F(\bar{Q})$	=	A general yield function
$F_1(N_x)$	=	Yield function containing N_x terms only
$F_2(N_\phi, M_\phi)$	=	Yield function containing N_ϕ and M_ϕ terms only
$[G]$	=	Constraint matrix
$2H$	=	Depth of the edge beam
h_1, h_2	=	Depths of the compression and tension zones respectively
$\bar{h}_1, \bar{h}_2$	=	Vertical distances of the centroids from the neutral axis of circular arcs above and below the neutral axis respectively
L	=	Span length of a shell or edge beam
LP	=	Linear Programming
M	=	Bending moment
M_p	=	Plastic moment capacity of a cylindrical shell cross section
M_ϕ, M_x	=	Bending moments per unit length of an axial section and a section perpendicular to the axis of a cylindrical shell, respectively
$M_{\phi x}, M_{x\phi}$	=	Twisting moments per unit length of an axial section and a section perpendicular to the axis of a cylindrical shell, respectively
$M_{\phi c}$	=	M_ϕ value at the crown of a cylindrical shell
M_o	=	$\frac{\sigma_c t^2}{4}$
M_r	=	Moment of resistance of a cylindrical shell cross section or edge beam
$\frac{M_\phi}{M_o}$	=	Reduced stress resultant
m_ϕ	=	Nondimensional transverse bending moment per unit length in a cylindrical shell (M_ϕ/M_o)

N	=	Axial force
N.A	=	Neutral axis
N_ϕ, N_x	=	Membrane (inplane) direct forces per unit length of a cylindrical shell
$N_{\phi x}, N_{x\phi}$	=	Membrane (inplane) shear forces per unit length of a cylindrical shell
$N_{\phi c}$	=	N_ϕ value at the crown of a cylindrical shell
N_{XA}, N_{XB}	=	Magnitudes of N_x in compression and tension zones respectively in lower bound solution 1
N_c	=	Compressive strength of a shell per unit length (σ_c t)
N_p	=	Maximum value of N_x in tension in lower bound solutions 2 and 3
N'_p	=	Strength of a cylindrical shell per unit length in tension
N_q	=	Maximum value of N_x in compression zone in lower bound solution 3
N_o	=	σ_c t
$\frac{N_x}{N_o}, \frac{N_\phi}{N_o}, \frac{N_{x\phi}}{S_o}$	=	Reduced stress resultants
n_ϕ	=	Nondimensional membrane direct force per unit length (N_ϕ/N_o)
$n_{\phi x}$	=	Nondimensional membrane shear force per unit length ($N_{\phi x}/S_o$)
P	=	Load
p	=	Intensity of gravity loading at collapse
p_1, p_2	=	Lower bounds in the nondimensional form due to N_c and $M_\phi - N_\phi$ yield criteria respectively
p_{L1}, p_{L2}	=	Lower bounds due to N_c and $M_\phi - N_\phi$ yield criteria respectively

p_o = Nondimensional lower bound

p_x, p_ϕ, p_z = Components of intensity of loading in x, ϕ and z directions

p_u = Upper bound

p_w = Intensity of working load

$\bar{Q}$ = Generalized stress vector

Q_x, Q_ϕ = Transverse shear forces per unit length in a cylindrical shell

q = Self weight of the edge beam per unit length

$\bar{q}$ = Generalized strain vector

R = Radius of a cylindrical shell

RC = Reinforced concrete

r = Nondimensional term $(\frac{R}{L})$

S_o = $0.1\ M_o$

s_o = $\frac{S_o}{M_o}$

t = Thickness of a cylindrical shell

$\bar{t}$ = Nondimensional term $(\frac{R}{t})$

V_v, V_h = Resultant vertical and horizontal components of forces acting on the edge beam

W_E = External work

W_I = Internal work

X, Y, Z = Coordinate axes

$\bar{x}$ = Nondimensional coordinate in x direction

x_j, x'_j, x''_j = Variables

$\{X\}, \{X\}^T$ = Column vector and its transpose respectively

$\{X_r\}$	=	Column vector containing the variables corresponding to the independent columns
$\{X_s\}$	=	Column vector containing the variables corresponding to the dependent columns
y_1	=	Vertical distance from the springing level to a point on the shell
Z_o	=	Objective function
z'	=	Vertical distance of a point on the shell either above or below the neutral axis from N.A
z'_1	=	Vertical distance of the point of maximum compression in compression zone from the neutral axis of a shell in lower bound solution 3
α_1	=	Upper limit for n_x in LP
α_2	=	Limit for $n_{x\phi}$ in LP
β	=	Angle defining the position of the neutral axis from the crown of a shell
β_1	=	Angle defining the position of maximum compression from the crown of a shell in the lower bound solution 3
δ	=	Arbitrary positive scalar
Δ	=	Displacement
Δx	=	Mesh size in x direction
$\Delta\bar{x}, \Delta\phi$	=	Nondimensional mesh sizes in $\bar{x}$ and ϕ directions
η	=	Factor defining the distribution of force in the edge beam
λ	=	Load factor
μ	=	Reduced steel ratio $\left(\frac{A_{st}\ \sigma_{sy}}{\sigma_c\ t}\right)$
σ_c	=	Compressive strength of concrete

σ_{sy}	=	Steel yield stress
ϕ	=	Angle between the normals at the crown and at any point on the cylindrical shell section
ϕ_o	=	Semicentral angle of a cylindrical shell
ψ	=	Factor defining the depth of steel from zero strain rate level in a reinforced concrete section

SYNOPSIS

LIMIT ANALYSIS OF REINFORCED CONCRETE CYLINDRICAL SHELL ROOFS

(A thesis submitted in partial fulfilment of the requirement for the degree of Doctor of Philosophy by K. Surayya to the Department of Civil Engineering, Indian Institute of Technology, Kanpur, June 1978)

A structure may be analysed either by elastic or limit analysis. The present study is focussed on the limit analysis of reinforced concrete shell roofs. Limit analysis is based on two fundamental theorems, viz.,(i) static (or lower bound) theorem and (ii) kinematic (or upper bound) theorem. These two theorems provide lower and upper bounds to the actual collapse load.

M.M. Fialkow proposed both lower and upper bound solutions for simply supported cylindrical shell roofs. Rigid plastic analysis with linear approximation to the Mises and Tresca yield criteria is adopted. N.A. Shoeb and W. C. Schnobrich developed elastic-plastic solution for cylindrical shell roofs using the Mises yield criterion. A.L.L. Baker considered circular reinforced concrete cylindrical shell as a cracked beam of curved cross section. M. Janas and A.R. Rzhanitsyn proposed upper bound solutions independently, for simply supported reinforced concrete cylindrical shell roofs.

C.S. Lin and A.C. Scordelis applied finite element technique to analyse reinforced concrete shells of general form.

The present study is concerned with the development of lower bound solutions for simply supported reinforced concrete circular cylindrical shell roofs subjected to uniform gravity loading. The longitudinal edges of the shells considered herein are either free or with edge beams.

A unique yield condition for a reinforced concrete cylindrical shell element is not available in the literature. Based on some assumptions two approximate yield criteria are proposed, viz.,(i) N_c – yield criterion and (ii) M_ϕ – N_ϕ yield criterion. The N_c-yield criterion states that the shell element yields if the longitudinal compression, N_x, reaches a value, N_c, which is equal to the strength of the shell in compression per unit length. The M_ϕ – N_ϕ yield criterion states that the shell element yields due to the interaction of bending moment, M_ϕ, and inplane direct force, N_ϕ, in the transverse direction. In this study yielding of the shell element is defined if at least one of the two yield conditions is satisfied. To simplify the analysis, the M_ϕ – N_ϕ yield criterion which is nonlinear is approximated as piece-wise linear.

In general a cylindrical shell element is subjected to inplane direct forces N_x and N_ϕ, inplane shears $N_{x\phi}$ and

$N_{\phi x}$, transverse shears Q_x and Q_ϕ, bending moments M_x and M_ϕ, and twisting moments $M_{x\phi}$ and $M_{\phi x}$. But for thin shells they are reduced to N_x, N_ϕ, $N_{x\phi}$, Q_x, Q_ϕ, M_x, M_ϕ and $M_{x\phi}$. In this study, the stress resultants Q_x, M_x and $M_{x\phi}$ are neglected. Since these stress resultants cannot be neglected for short shells, these solutions are not applicable to this class of shells. The equilibrium equations are reduced to four, containing five unknown stress resultants.

Using static theorem lower bound solutions are developed. Since the number of stress resultants are more than the equilibrium equations by one, N_x distribution is assumed in all the solutions. Its magnitude is determined from the static equilibrium conditions of the shell. Knowing N_x, other stress resultants are solved using equilibrium equations and boundary conditions for the stress resultants. Three solutions are developed with three different distributions for N_x. In the first solution, N_x is taken as uniform both in tension and compression zones across the section. In the second solution, N_x varies linearly in tension zone and uniform in compression zone. In the third solution, parabolic distribution is taken in the compression zone keeping linear variation in tension zone. In the solution three, β_1, the angle defining the position of maximum longitudinal compression is kept as a variable to obtain the optimum lower bounds.

For shells with edge beams, the longitudinal force developed in the edge beam is taken as trapezoidal.

Shells with different geometric parameters are analysed by both elastic (ASCE, Manual No. 31) and by the proposed lower bound solutions. Distribution of stress resultants at critical sections are presented for comparative study. Two design examples are also presented by elastic analysis for service load and by the proposed limit analysis for a load factor of 2. The economy of the materials is discussed.

A numerical technique is also developed. In this, the problem is formulated as a linear programming problem and the revised simplex method is used to solve it. The shell is considered as a grid of discrete points. Finite difference scheme is used for equilibrium equations. The load is taken as an objective function to be optimized and the stress resultants at the grid points are taken as variables. The constraints are formulated in terms of these variables at each grid point to satisfy the yield condition. The load is optimized using the simplex method and the corresponding variables are evaluated. The distribution of stress resultants at critical sectionsis also presented.

The thesis concludes with the discussion of the results obtained.

CHAPTER 1

INTRODUCTION

1.1 General

The word 'shell' primarily means any curved shape/surface enclosing/covering certain volume/area. It has different connotations for persons in varied environment. The shells we find around in abundance are either God's creation or man-made. The hard outside covering of a seed, a fruit or an animal, the hard calcareous envelope of a bird's egg fall in the former category. An explosive projectile used in a cannon or a mortar may be categorised in the latter variety. If we study the nature in depth it will at once be realised that microscopic cells in the living beings, root sections of birds' feathers and the bamboos used in building construction from the times immemorial are also shell shaped. The egg shells range widely in size and shape.

Man by observing nature critically, realised that a given quantum of material if laid out judiciously can possess tremendous load carrying capabilities. The obvious shape is that of a shell. Now virtually every item of the most modern industrial equipment, be it aero-space, building, nuclear, marine or petro-chemical, has shell as a structural component.

Although all structures are invariably three-dimensional, for purposes of analysis they are characterised as one-dimensional, two-dimensional or three-dimensional, depending upon the relative ratios of their principal dimensions. In that sense, shells are two-dimensional (thickness is small in comparison with the other two principal dimensions) and are represented by the middle surface. Thus an arbitrary point of the middle surface of a shell may be defined by specifying two scalar components. The complete geometry of a shell is defined by describing the middle surface and the thickness.

As defined earlier, a shell is a curved surface. It is called thin if the maximum value of the ratio of the thickness to the radius of curvature is very small in comparison with unity. The shells may be curved in one or two principal directions and are termed singly or doubly curved respectively. A cylindrical shell falls in the former category while a spherical one falls in the latter.

In this thesis a cylindrical shell is defined as a curved slab (singly curved shell) cut out from a cylinder. The slab is thus bounded by two straight longitudinal edges parallel to the axis of the cylinder and by two curved transverse edges in planes perpendicular to the axis. The longitudinal edges are either free or stiffened with beams.

Shells may be made of different materials depending upon the functional, strength and weight requirements. In building industry most of the shells are made of reinforced concrete. Reinforced concrete shell structures possess many of the qualities of an ideal structure. They require a minimum amount of material and permit coverage of large areas with out intermediate supports. Their form offer virtually unlimited possibilities for aesthetic and architectural considerations. Cylindrical shells are also popularly adopted for their simplicity in form work.

Rigorous and approximate analytical methods of analysis and design are readily available in literature. However these are mostly confined to elastic theory. It has long been recognised that reinforced concrete shells are usually too heavily loaded to remain in the range of validity of linear elastic theory. In addition this method of analysis has one more limitation that the behaviour of a structure can be studied within a specified limit of working stress. The factor of safety here is with respect to the ultimate stresses of the materials but not with respect to ultimate or collapse load of a structure. In practice elastically designed structures are conservative resulting in wastage of precious materials. Hence rational methods of analysis have come into vogue taking the collapse loads into consideration. These are known as plastic limit or simply limit

analysis methods, which take into account the true factor of safety against collapse. The ratio of collapse load to working load is known as load factor. Every method has its own limitations. The limit analysis methods are no exception. The behaviour of the structure at working load cannot be predicted by these methods. The present study is confined to the limit analysis of reinforced concrete cylindrical shell roofs. In the following paragraph a comprehensive literature survey is presented.

1.2 Selective Literature Survey

The previous investigations on the limit analysis of shells have been restricted **mainly to** axisymmetric shells (1,2,3,4)* except for a few on cylindrical shell roofs. Fialkow (5) proposed both lower and upper bound solutions for simply supported long cylindrical shell roofs with free longitudinal edges under radial loading. An insight into the limit analysis of cylindrical shell roofs is given. The material is idealised as a rigid plastic material obeying the Tresca or linearized Mises yield criterion. Shoeb and Schnobrich (6) presented a solution for elastic-plastic analysis of cylindrical shell roofs with a discrete mathematical model using the Prandtl-Reuss incremental theory of plasticity together with Mises yield condition. Wagner & Yang(7)

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* Numbers in the brackets indicate the reference number.

obtained elasto-plastic solution for the limit analysis of cylindrical shells using finite element technique. The material obeys the Mises yield condition.

Thus the limit analysis is mainly confined to materials obeying the Mises and Tresca yield criteria.

Very few upper bound solutions are available for rotationally symmetric reinforced concrete shells like spherical domes and conical shells (8,9). Simple failure mechanisms were assumed and the collapse loads were arrived at by equating the external work done by the load during deformation to the internal energy of dissipation. Lower and upper bound solutions are available for axisymmetric shells like water tanks (10). Baker (11,12) determined the ultimate load of reinforced concrete cylindrical shell roofs. The shell was considered as a cracked beam of thin curved cross section in the longitudinal direction and as an arch in the transverse direction. This analysis was not based on the fundamentals of limit analysis but on the principle of cracked reinforced concrete section. Rzhanitsyn (13) developed a simplified kinematic method for shallow shells. Sawczuk (14) presented upper bound solutions for reinforced concrete cylindrical shells resting on rigid supports with different edge conditions. These were based on collapse mechanisms suggested from experiments. Janas (15) proposed limit analysis of

nonsymmetrical shells by generalized yield line method. Rzhanitsyn (16) introduced a solution to determine the bearing capacity of reinforced concrete cylindrical shell roof supported by end diaphragms. The extremum principle of the kinematic method of limit equilibrium was used as the basis for this analysis. Lin and Scordelis (17,18) applied the finite element technique to analyse reinforced concrete shells of general form. The analysis includes the load-deformation response and crack propagation and determines the internal concrete and steel stresses.

Besides, some experimental results of reinforced concrete shell roofs are also available in the literature. Bouma et al. (19) conducted a series of tests on cylindrical shells with edge beams to demonstrate the ultimate load behaviour of shells. A series of eleven shells with the same cross section but of different spans were tested. The load factor ranged from 2.75 to 4.50. Enami (20) conducted a number of experiments on reinforced concrete folded plate structures and based on the failure patterns presented a series of mechanisms. Harris and White (21) conducted experiments on small scale models to investigate elastic and inelastic behaviour of reinforced concrete shell roof structures.

From the forgoing it is obvious that most of the structures are confined to arriving at upper bound solutions

which are inherently unsafe in the absence of a correct collapse mechanism. It is imperative that methods of analysis based on the lower bound theorem are more appropriate. Hence this study is confined to lower bound methods of analysis.

1.3 Object and Scope of Present Work

A few upper bound solutions are available to determine the ultimate load capacity of cylindrical shell roofs. The major objective of the present work is to develop solutions for simply supported reinforced concrete cylindrical shell roofs based on 'static theorem'. The cylindrical shell roofs analysed in this study are classified into two groups, viz., (i) cylindrical shell roofs with free longitudinal edges and (ii) cylindrical shell roofs with edge beams along the longitudinal edges.

Presently no unique yield condition is available in the literature for reinforced concrete cylindrical shell element. With this in view two independent yield conditions are proposed, viz., (i) 'L_c - yield criterion' - yielding due to maximum longitudinal compression and (ii) 'M_ϕ - N_ϕ yield criterion' - yielding in the transverse direction due to the interaction of the transverse moment M_ϕ and the inplane direct force N_ϕ.

Three lower bound solutions are presented for cylindrical shell roofs with free longitudinal edges. The

variation of the ultimate load in nondimensional form to the ratio of span L to the radius R is indicated by way of graphs. Two shells are analysed with different geometric parameters indicating the distribution of stress resultants at collapse. Two shells with the same geometric parameters are designed for a given load factor by the proposed lower bound solutions. Elastic and plastic design methods are compared with material economy in view.

For cylindrical shells with edge beams the following aspects are studied. Three lower bound solutions are developed. Two shells of different geometric parameters are analysed to indicate the distribution of stress resultants at collapse. Two shells with same geometric parameters are designed for a given load factor by the proposed lower bound solutions. The economy of the materials by both elastic and plastic methods is discussed.

The ultimate load calculated from the present lower bound solutions is compared with the ultimate load determined from the upper bound solution given in the literature (22). It is found that both values are the same if the shell fails as per the N_c-yield criterion. Hence the solution is exact when the shell obeys the N_c-yield criterion.

A numerical technique is also developed to arrive at the lower bound solutions. It is formulated as a linear

programming (LP) problem and revised simplex method is used to solve it. The shell is considered as a grid of discrete points. Finite difference scheme is used to replace differential equations of equilibrium by algebraic equations. The load is taken as an objective function and the stress resultants are taken as variables. The constraints are formulated in terms of these variables at each grid point so that the yield condition is not violated. The load is optimized using the simplex method and the corresponding variables are evaluated. The distribution of stress resultants is presented.

The thesis concludes with the discussion of the results obtained.

CHAPTER 2

BACKGROUND THEORY

2.1 General

The purpose of structural analysis is to study or predict the response or behaviour of a loaded structure. It is based on four requirements viz., the equilibrium, the condition of compatibility, the stress-strain relationship of the material used and the known boundary conditions. The equilibrium equations specify the relationship between the internal stress resultants and the external loads applied. The relation between displacements and strains are termed compatibility equations. The boundary conditions may be static (bending moment, shear force etc.) or kinematic (deflection, rotation etc.) constraints on the structure and are usually prescribed over certain domain. The stress-strain relations are determined by simple tests. If the foregoing four conditions are satisfied by a solution at every point in a body, that solution is said to be exact and yields correct stresses, strains and displacements in the body. Sometimes it is not possible to satisfy all the four conditions, one or more of them, may be relaxed. A solution satisfying less than these four conditions is approximate. Depending on the range of mechanical behaviour, analysis may be broadly classified as elastic or plastic.

If the relationship between the stresses and strains is a single valued (unique) and does not depend on the stress history (no effects of viscosity and hystersis) then the material is termed elastic. When the foregoing relation is linear then the material is linearly elastic (The material is said to obey generalised Hooke's law). Elastic analysis predicts the behaviour at working loads while plastic analysis does at failure. The stress-strain relation for a rigid-perfectly plastic material is termed flow rule. Because of the non-unique nature of the stress-strain relation for a plastic material, complete solutions are difficult to get for complex structures. As a result approximate solutions known as bounds on the correct collapse loads are usually found in literature.

In this chapter an attempt has been made to present the equilibrium equations of a cylindrical shell element , limit analysis theorems and various yield conditions.

2.2 Equilibrium Equations

A cylindrical shell may be defined as a curved slab which has been cut out from a full cylinder. The slab is bounded by two straight 'longitudinal edges' parallel to the axis of the cylinder and by two curved 'transverse edges' in planes perpendicular to the axis. The slab is therefore, curved in only one direction. When the curvature is constant, the cylindrical shell is said to be circular.

The equilibrium equations are derived based on the following assumptions : i) The shell is thin. ii) The deflections are small at the instant of collapse. With the assumption (ii), the equilibrium equations are derived based on the undeformed geometry of the shell element.

Fig. 2.1a shows a typical shell, its geometry and also the positive directions of the coordinate axes. Origin is taken at the apex of the shell directrix at left hand support. The coordinate x is measured along the crown generator, the coordinate y along the directrix, the coordinate z along the inward normal. The forces and moments acting on a shell per unit length are usually known as the stress resultants. In general, a cylindrical shell element is under the action of inplane direct forces N_x and N_ϕ, inplane shears $N_{x\phi}$ and $N_{\phi x}$, transverse shears Q_x and Q_ϕ, bending moments M_x and M_ϕ and twisting moments $M_{x\phi}$ and $M_{\phi x}$. The stress resultants and sign convention adopted are shown in Figs. 2.1b and c. p_x, p_ϕ and p_z denote the components of external load applied per unit area in x, y and z directions respectively. The equilibrium equations for the shell element are given as (23)

$$R \frac{\partial N_x}{\partial x} + \frac{\partial N_{\phi x}}{\partial \phi} + p_x R = 0 \quad , \qquad \ldots (2.1)$$

$$\frac{\partial N_\phi}{\partial \phi} + R \frac{\partial N_{x\phi}}{\partial x} - Q_\phi + p_\phi R = 0 \quad , \qquad \ldots (2.2)$$

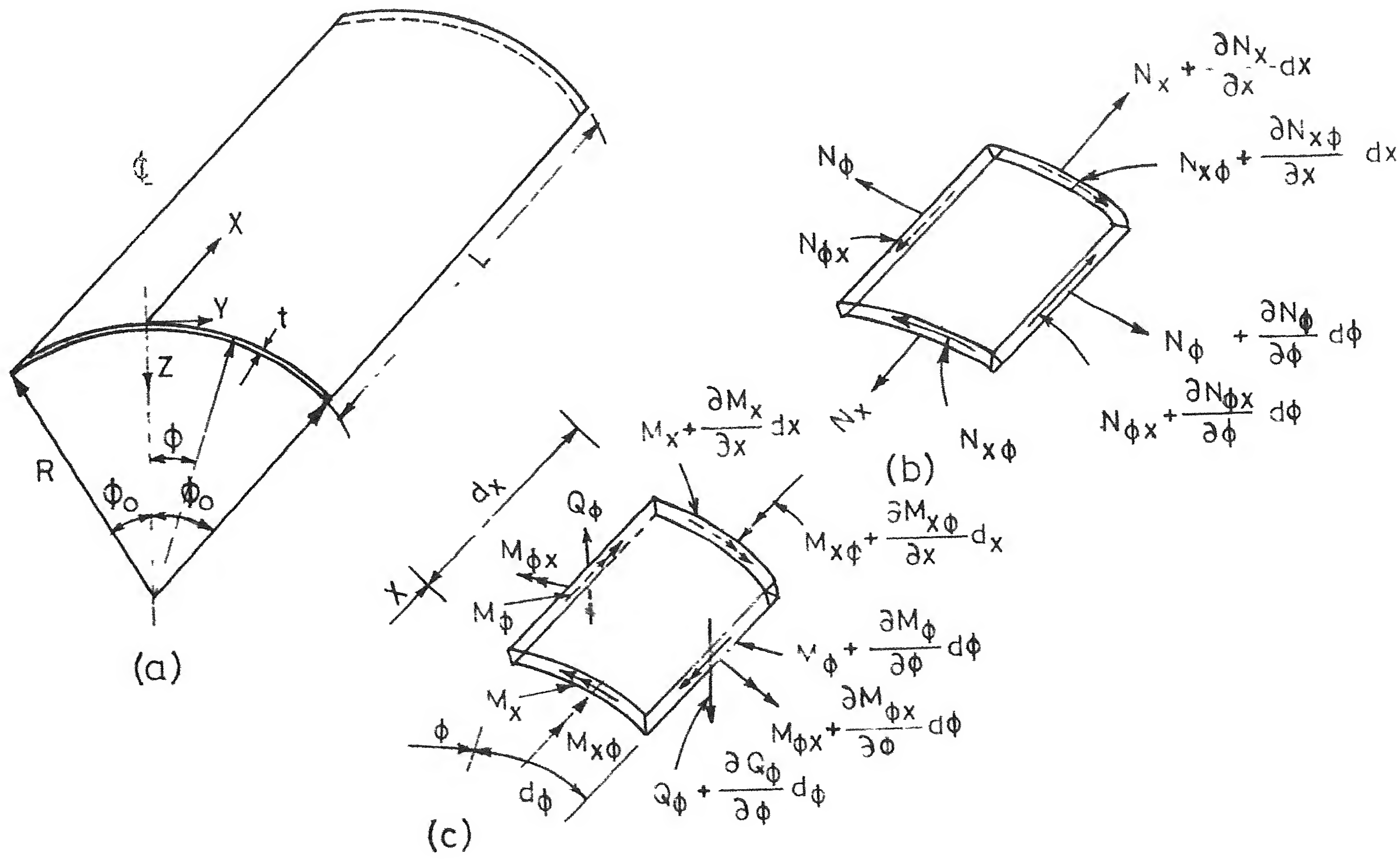


FIG.2.1 COORDINATE SYSTEM AND SIGN CONVENTION

$$R \frac{\partial Q_x}{\partial x} + \frac{\partial Q_\phi}{\partial \phi} + N_\phi + p_z R = 0 \ , \qquad \ldots (2.3)$$

$$- \frac{\partial M_\phi}{\partial \phi} + R \frac{\partial M_{x\phi}}{\partial x} + Q_\phi R = 0 \ , \qquad \ldots (2.4)$$

$$R \frac{\partial M_x}{\partial x} - \frac{\partial M_{\phi x}}{\partial \phi} + Q_x R = 0, \qquad \ldots (2.5)$$

where R is the radius of curvature. For thin shells it can be readily shown that $N_{x\phi} = N_{\phi x}$ and $M_{x\phi} = M_{\phi x}$. In all, there are five equations of equilibrium and eight unknown stress resultants. Thus equilibrium equations alone are not sufficient for solving stress resultants. To reduce the complexity of the problem, certain simplifications are made based on the structural action.

In the literature some numerical studies as well as experimental results show that the state of stress in a shell depends essentially on the ratio of the span L to the chord width B (24). Depending on the $\frac{L}{B}$ ratio it is possible to neglect certain stress resultants. Accordingly cylindrical shells are classified as short ($\frac{L}{B} < 1$), medium ($1 < \frac{L}{B} < 3$) and long ($\frac{L}{B} > 3$).

The short cylindrical shells are characterized by the appearance of all stress resultants that occur in a shell. In the case of medium length cylindrical shells, theoretical and experimental studies have shown that the state of stress

is influenced primarily by the transverse bending moment M_ϕ, while the effects of twisting moments $M_{x\phi}$, $M_{\phi x}$, the bending moment M_x and shear force Q_x are negligible. Longitudinal shells can be treated as thin walled rods, based on the hypothesis that the cross section is not deformed.

The present study is mainly concerned with medium cylindrical shells. It can be applied to long cylindrical shells as well. In this thesis no distinction is made between medium and long cylindrical shells.

Neglecting the stress resultants M_x, Q_x and $M_{x\phi}$, the equilibrium equations (2.1) through (2.5) reduce to

$$R \frac{\partial N_x}{\partial x} + \frac{\partial N_{\phi x}}{\partial \phi} + p_x R = 0 , \qquad \ldots (2.6)$$

$$\frac{\partial N_\phi}{\partial \phi} + R \frac{\partial N_{x\phi}}{\partial x} - Q_\phi + p_\phi R = 0 , \qquad \ldots (2.7)$$

$$\frac{\partial Q_\phi}{\partial \phi} + N_\phi + p_z R = 0 , \qquad \ldots (2.8)$$

$$-\frac{1}{R} \frac{\partial M_\phi}{\partial \phi} + Q_\phi = 0 . \qquad \ldots (2.9)$$

The shell is subjected to gravity loading only. If p is the intensity of loading, then its components are given by

$$\begin{aligned} p_x &= 0 , \\ p_\phi &= p \sin\phi , \\ p_z &= p \cos\phi . \end{aligned} \qquad \ldots (2.10)$$

Differentiating Eqs. (2.6) and (2.7) with respect to x and ϕ respectively and using the relations (2.10) yields

$$R \frac{\partial^2 N_x}{\partial x^2} + \frac{\partial^2 N_{\phi x}}{\partial x \partial \phi} = 0 , \qquad \ldots (2.11)$$

and

$$\frac{\partial^2 N_\phi}{\partial \phi^2} + R \frac{\partial^2 N_{x\phi}}{\partial x \partial \phi} - \frac{\partial Q_\phi}{\partial \phi} + p\, R \cos \phi = 0. \qquad \ldots (2.12)$$

Eq. (2.12) is simplified using Eqs. (2.8) and (2.11) as

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} - R^2 \frac{\partial^2 N_x}{\partial x^2} + 2\, p\, R \cos \phi = 0. \qquad \ldots (2.13)$$

Eliminating Q_ϕ from Eq. (2.9) and substituting in Eq. (2.8) for Q_ϕ one gets

$$N_\phi + \frac{1}{R} \frac{\partial^2 N_\phi}{\partial \phi^2} + p\, R \cos \phi = 0. \qquad \ldots (2.14)$$

Thus the equilibriums equations (2.6) through (2.9) are reduced to following equations :

$$R \frac{\partial N_x}{\partial x} + \frac{\partial N_{\phi x}}{\partial} = 0, \qquad \ldots (2.15)$$

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} - R^2 \frac{\partial^2 N_x}{\partial x^2} + 2\, p\, R \cos \phi = 0, \qquad \ldots (2.16)$$

$$N_\phi + \frac{1}{R} \frac{\partial^2 N_\phi}{\partial \phi^2} + p\, R \cos \phi = 0 , \qquad \ldots (2.17)$$

$$- \frac{\partial N_\phi}{\partial \phi} + Q_\phi R = 0. \qquad \ldots (2.18)$$

The modified form of equilibrium equations (2.15) through (2.18) are used in the analysis.

2.3 Basic Concepts of Plasticity

2.3.1 Plasticity

Engineering plasticity is concerned mainly with the practical applications of the theory of plasticity. When a solid is subjected to external forces, it usually deforms. If the body does not regain its original shape after the external forces have been removed, then the body is said to have undergone plastic deformation, or the plastic flow of the material in the body has occured. Thus plasticity deals with the behaviour of deformable solids, which undergo permanent deformation under the action of external influences. Only time independent deformations are considered here.

2.3.2. Limit Analysis

Plastic analysis and limit analysis are synonymous terms. Plastic analysis is a branch of structural analysis dealing with the structures loaded into the plastic range. Only those structures whose relationship between the generalised stresses and generalized strains can be idealized by either elastic-perfectly plastic or rigid-perfectly plastic behaviour are considered here.

The object of plastic limit analysis is the calculation of collapse loads, at which the structures continue to deform while the loads remain constant, in other words unrestricted plastic flow occurs in the structure. In this thesis only the limit approach to design and analysis is adopted, i.e., the conditions at the point of impending collapse are considered in determining the collapse loads. As the analysis and design are based on limit approach, they are termed as limit analysis and limit design respectively. The object of limit analysis may also be stated as determining the safety against collapse. Limit analysis is based on three fundamental theorems viz., the lower bound (static) theorem, the upper bound (kinematic) theorem and the uniqueness theorem. In the limit analysis, the safety factor usually known as load factor is the ratio of collapse load to the design or working load. Although the calculated safety factor in limit analysis is very important, the disadvantage of limit analysis is that no information is available on the behaviour of a structure at or below working loads. The ultimate object of the limit analysis is to find safety factor against collapse of a structure, but this may not be feasible in all the cases. In such cases, upper and lower bounds on the safety factor should be determined. In this thesis both analytical and numerical methods are considered with reference to reinforced concrete cylindrical shell roofs. Emphasis is made on the

determination of lower bounds to the collapse load using lower bound theorem. In limit analysis the structure is fully specified and the maximum safe load according to some criterion is to be determined.

2.3.3 Terminology

In this thesis the terminology and notation closely follows Save and Massonnet (22).

Perfectly Plastic Material :

In this material, strains are independent of time and it does not exhibit strain hardening properties. It is capable of flowing indefinitely under constant stress when the stress reaches a certain magnitude known as yield stress. A material characterized by these properties is termed perfectly plastic.

Elastic-Plastic and Rigid-Plastic Bodies :

Both the idealizations of elastic perfectly plastic and rigid perfectly plastic material are appropriate for the purposes of limit analysis. These are shown in Figs. 2.2a and b respectively. When the elastic-perfectly plastic idealization is used, the limit state corresponds to the incipient plastic flow. When using the rigid-perfectly plastic idealization, the rigid parts of the body prevent all deformations upto the on set of unrestrained plastic flow at the limit load.

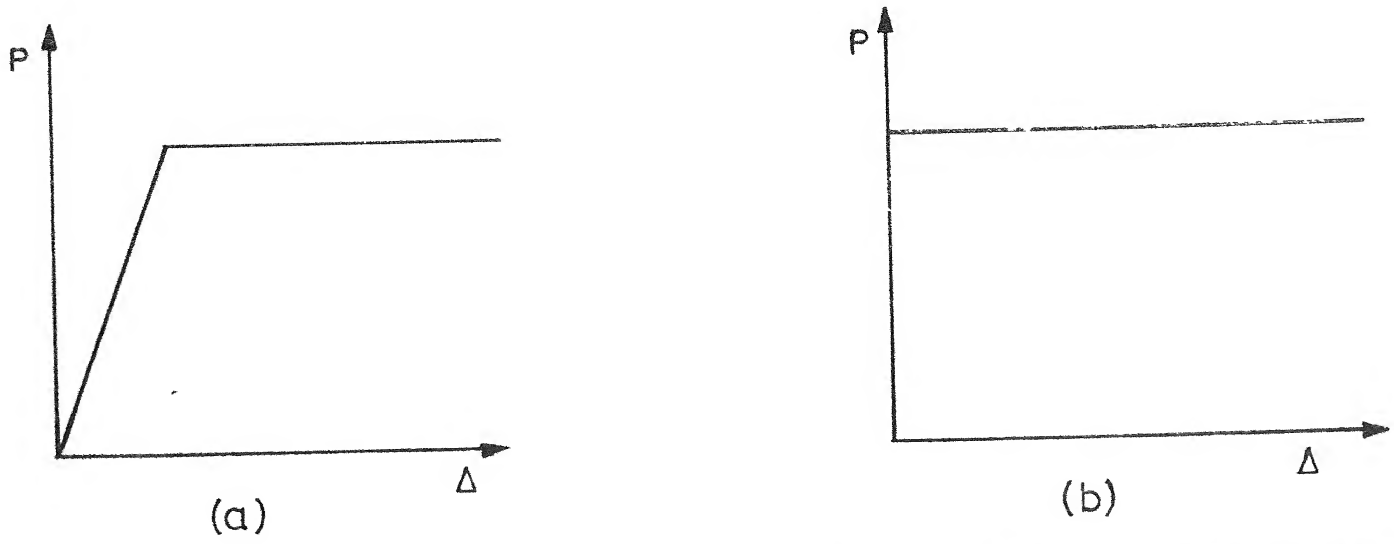


FIG.2.2 (a) ELASTIC-PLASTIC (b) RIGID-PLASTIC IDEALIZATIONS

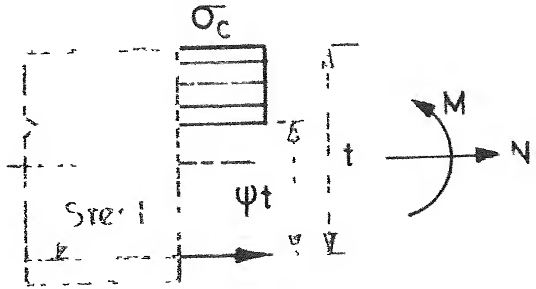


FIG.2.3 M-N INTERACTION

Collapse (limit) Load :

Let a structure be subjected to a system of loads that are increased in proportion, starting from zero. When the load reaches a particular magnitude, the deformations of the structure increase beyond all bounds under a constant value of load intensity. The corresponding value of the load is defined as collapse or limit load.

Proportional Loading :

If all the forces acting on a structure can be expressed in proportion to one parameter, then the loading is said to be proportional.

2.3.4 Physical Conditions

These are also known as the constitutive equations or stress-strain relations in the theory of perfectly plastic materials. They consist of two parts, the yield condition and flow rule.

Yield Condition :

A yield condition is a hypothesis concerning the limit of elasticity under any possible combination of generalized stresses. It is a function that relates the generalized stresses at yield and is given by

$$F(\bar{Q}) = c_1 , \qquad \ldots (2.19)$$

where $\overline{Q}$ is the generalized stress vector, and c_1 is a material constant, which characterizes the yield.

The sign of the function F adopted is such that no yield occurs when

$$F(\overline{Q}) < c_1 \, . \qquad \ldots (2.20)$$

The yield occurs only when equation (2.19), is satisfied and the combinations of stresses corresponding to

$$F(\overline{Q}) > c_1 \qquad \ldots (2.21)$$

are inadmissible.

The point, curve or surface corresponding to equation (2.19) is known as the yield point, the yield curve or the yield surface respectively in one-two-and n - ($n \geq 3$) dimensional stress space represented by the components of the stress vector $\overline{Q}$. If $n > 3$, the surface is a hyper-surface. The yield surface can be shown to be always convex.

Flow Rule :

The flow rule expresses the ratio of plastic strain components during yielding. The flow rule is assumed in such way that the generalized strain vector $\overline{q}$ is given by,

$$\overline{q}_i = \delta \frac{\partial F}{\partial \overline{Q}_i} \qquad i = 1,2,\ldots n \qquad \ldots (2.22)$$

where δ is an arbitrary factor of proportionality. As the yield surface is considered to be convex and since the plastic

work done is always nonnegative

$$\delta \geq 0 .$$

The Eq. (2.22) implies that the strain vector is normal to the yield surface given by Eq. (2.19). Hence the flow rule is sometimes referred to as the 'normality flow rule'.

2.3.5 Theorems of Limit Analysis

Statically Admissible Stress Field :

The stresses are in internal equilibrium, in equilibrium with the applied load P, and satisfy the static boundary conditions. The stresses should nowhere violate yield inequality. A stress field of this kind is called statically admissible.

Kinematically Admissible Velocity (Displacement) Field :

Any displacement field compatible with the geometric (kinematic) boundary conditions and certain continuity conditions, is said to be a kinematically admissible displacement field.

The rigid plastic model (Fig. 2.2b) and elastic-plastic model (Fig. 2.2a) yield the same load capacity values.

Lowerbound (Statical) Theorem :

Any load P^- corresponding to a statically admissible stress field, is smaller than or at most equal to the limit load P_0

$$P^- \leq P_0 .$$

Upperbound (Kinematic) Theorem :

Any load P^+ corresponding to a kinematically admissible displacement field is larger than or at least equal to the limit load P_0

$$P_0 \leq P^+$$

Uniqueness Theorem :

The load capacity is unique if derived from a statically admissible stress field for which a corresponding kinematically admissible displacement field exists.

This theorem may thus be regarded as a combination of first two theorems.

2.3.6 Yield Criteria

In the literature the Tresca and von Mises criteria are applied to metallic plates and shells while the square yield criterion is applied to reinforced concrete slab element in certain situations. A special yield condition (22) is available for a general slab element which is under the action of the two bending moments and the twisting moment.

The yield condition (2.19) for a cylindrical shell element under consideration may be expressed as

$$F(N_x, M_{x\phi}, N_\phi, M_\phi, Q_\phi) = c_1 \qquad \dots (2.23)$$

Yield condition is not available in the literature for reinforced concrete cylindrical shell element under the action of all the stress resultants. In the absence of a definite yield condition, the following assumptions are made to arrive at an approximate yield criterion.

(1) There is only limited interaction between the stress resultants i.e., there is no interaction between the stress resultants acting in the x and ϕ directions at any point of the shell.

(2) The influence of transverse shear Q_ϕ on the yield condition is neglected.

(3) The effect of membrane shear $N_{x\phi}$ is decoupled from other stress resultants and also it is assumed that the shell does not fail due to insufficient reinforcement provided to take diagonal tension developed due to $N_{x\phi}$.

(4) The collapse of the shell is not initiated due to the yielding of longitudinal steel provided to resist N_x, in tension zone and hence the shell fails due to longitudinal compression only.

(5) Failure of the shell due to buckling will not occur.

With these foregoing assumptions, the Eq. (2.23) may be split into the following two independent relations as

$$F_1 (N_x) = c_2 \qquad \dots (2.24)$$

and

$$F_3(N_\phi, M_\phi) = c_3 \qquad \ldots (2.25)$$

where c_2 and c_3 are material constants.

If at least one of the two inequalities becomes an equality, then the shell element is in a state of fully plastic condition.

N_c-yield Criterion :

As a consequence of the assumption 4 listed earlier, the Eq. (2.24) is valid in compression zone only and is given by

$$-N_c = N_x \qquad \ldots (2.26)$$

where $N_c = \sigma_c t$, is the strength of the shell in compression per unit length. σ_c and t are compressive strength of concrete and thickness of shell respectively. Henceforth this yield condition is termed as N_c-yield criterion.

M_ϕ-N_ϕ Yield Criterion :

The condition described by Eq. (2.25) gives the failure of a reinforced concrete section under the interaction of an axial force N, and a bending moment M.

To obtain the M-N interaction curve (22), in a yielding cross section it is assumed that concrete is yielding in compression at a constant stress $-\sigma_c$ above the zero strain rate level, and below that level, the concrete has cracked and the steel yields in tension at a constant stress σ_{sy}.

If A_{st} denotes the cross-sectional area of the reinforcement per unit length, then (Fig. 2.3) the bending moment M and the axial force N are given by

$$M = \frac{\sigma_c t^2}{2} (1 - \psi),$$

and ... (2.27)

$$N = A_{st} \sigma_{sy} - \sigma_c t (1 - \psi).$$

Denoting $M_o = \frac{\sigma_c t^2}{4}$ and $N_o = \sigma_c t$, M and N can be rewritten as

$$M = 2M_o(1 - \psi)$$
... (2.8)
$$N = N_o(\mu - 1 + \psi) = \frac{4M_o}{t} (\mu - 1 + \psi)$$

where $\mu = \frac{A_{st} \sigma_{sy}}{\sigma_c t}$, is the "reduced reinforcement ratio".

Elimination of the parameter ψ from Eq. (2.28) results in

$$\frac{M}{M_o} = 2 \left[\mu (2 - \mu) - 2 \frac{N}{N_o} (1 - \mu) - \left(\frac{N}{N_o}\right)^2 \right] \quad \text{... (2.29)}$$

The two yield conditions described by the Eqs. (2.26) and (2.29) can be represented in three dimensional stress space N_x, N_ϕ and M_ϕ as shown in Fig. 2.4a.

For a given thickness of a shell, the M_ϕ-N_ϕ interaction curve depends on the quantity of transverse steel. In this analysis the value of A_{st} is assumed such that $\mu = 0.236$.

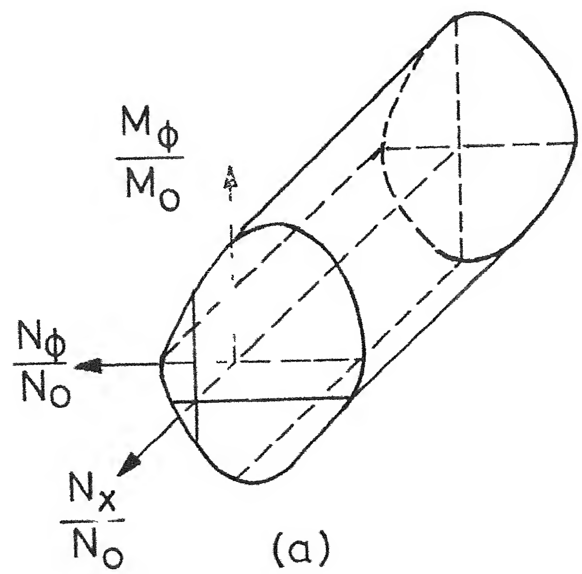


FIG. 2.4 YIELD SURFACE

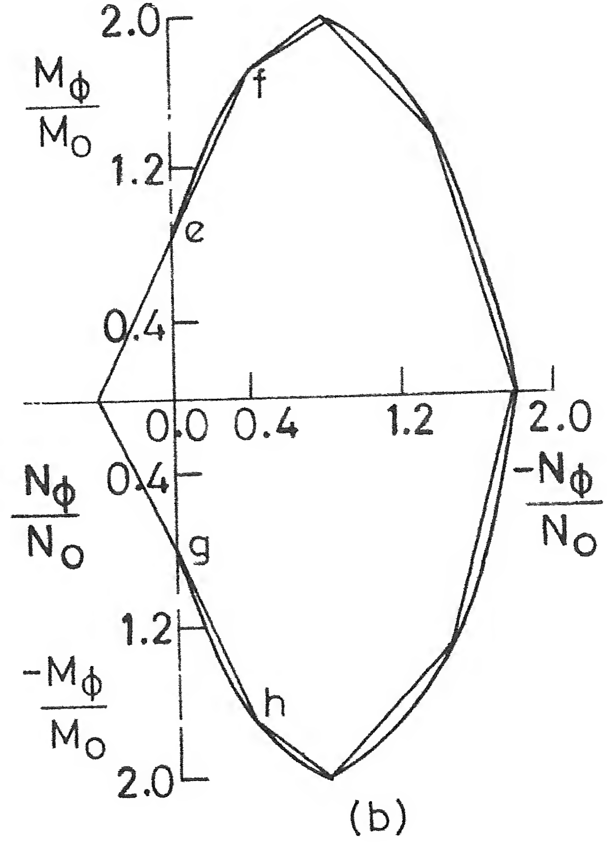


FIG 2.4 $\frac{M_\phi}{M_o} - \frac{N_\phi}{N_o}$ INTERACTION CURVE

This quantity is the amount of steel required for a balanced rectangular section according to ultimate strength design under pure bending as per I.S. Code (25). To reduce the complexity of the problem, the yield surface is linearized suitably in the M_ϕ-N_ϕ plane as shown in Fig.2.4b. As N_ϕ is always compressive and for the practical range of shells, the condition is governed by the straight lines ef or gh of Fig. 2.4b and are given as

$$\left| \frac{M_\phi}{M_o} \right| = b + c \left| \frac{N_\phi}{N_o} \right| \qquad \ldots (2.30)$$

For the value of $\mu = 0.236$, the absolute values of b and c are evaluated as 0.833 and 2.25 respectively. Hence this yield condition is termed as M_ϕ-N_ϕ yield criterion.

The yield criteria described by Eqs. (2.26) and (2.30) are adopted in this thesis.

CHAPTER 3

LIMIT ANALYSIS OF CYLINDRICAL SHELL ROOFS WITH FREE LONGITUDINAL EDGES

3.1 General

As the solutions based upon the static theorem of limit analysis are always safe, three solutions are developed for simply supported reinforced concrete (RC) circular cylindrical shell roofs subjected to gravity loading. The load is uniformly distributed. Such a loading is considered because the dead weight (including self weight) of shell is a major portion of the total load acting on the shell. The superimposed loads may also be considered as gravity loads depending upon the nature of the load. In such situations as they can be approximated as gravity loads the analysis becomes simpler.

The general problem of limit analysis, as already mentioned, is the determination of the load carrying capacity of a structure when the material just starts flowing. As it is difficult to get the exact loads (identical lower and upper bounds) the bounds are found. Of these, the solutions based on the upper bound are unsafe if the correct flow mechanism is not visualised.

It is well known that the upper bound solutions yield only collapse loads. Since the distribution of forces is not

known, economic distribution of steel is not possible as the distribution of steel is preassigned, where as in the case of lower bound solutions distribution of stress resultants is known throughout the shell, hence steel can be proportioned according to the requirements.

Two shells one each in the long and medium category of shells with different geometric parameters are analysed herein. The same are also analysed by elastic method when the shells are subjected to working loads. The distribution of stress resultants are shown graphically in both the cases for the purpose of comparison.

Later two shells are designed by both elastic and the proposed plastic methods. The relative economy of the materials used is also discussed.

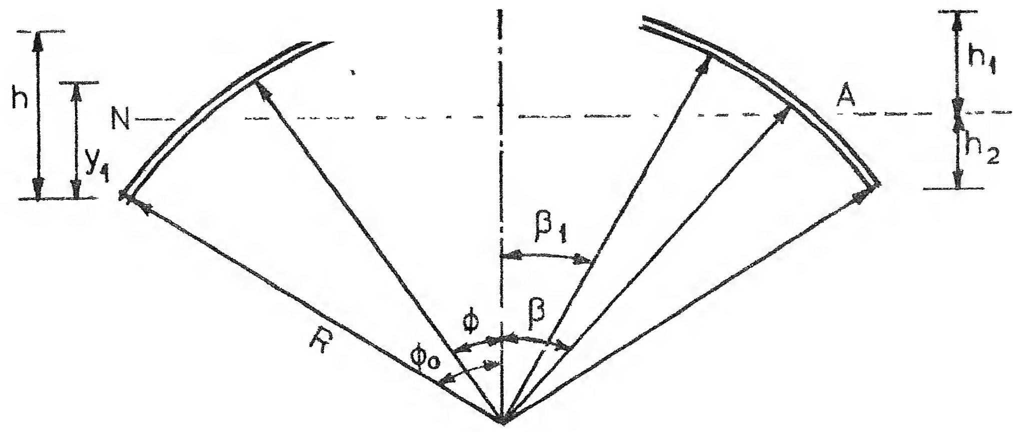
3.2 Lower Bound Solutions

The lower bound solutions are developed using static theorem. According to this theorem, any stress field satisfying the specified boundary conditions, that is in equilibrium with the applied loads and no where violates the yield condition is a lower bound to the actual collapse load. In view of this statement it is possible to construct stress fields which provide good lower bounds for the collapse load of a given structure. Thus the static method consists of

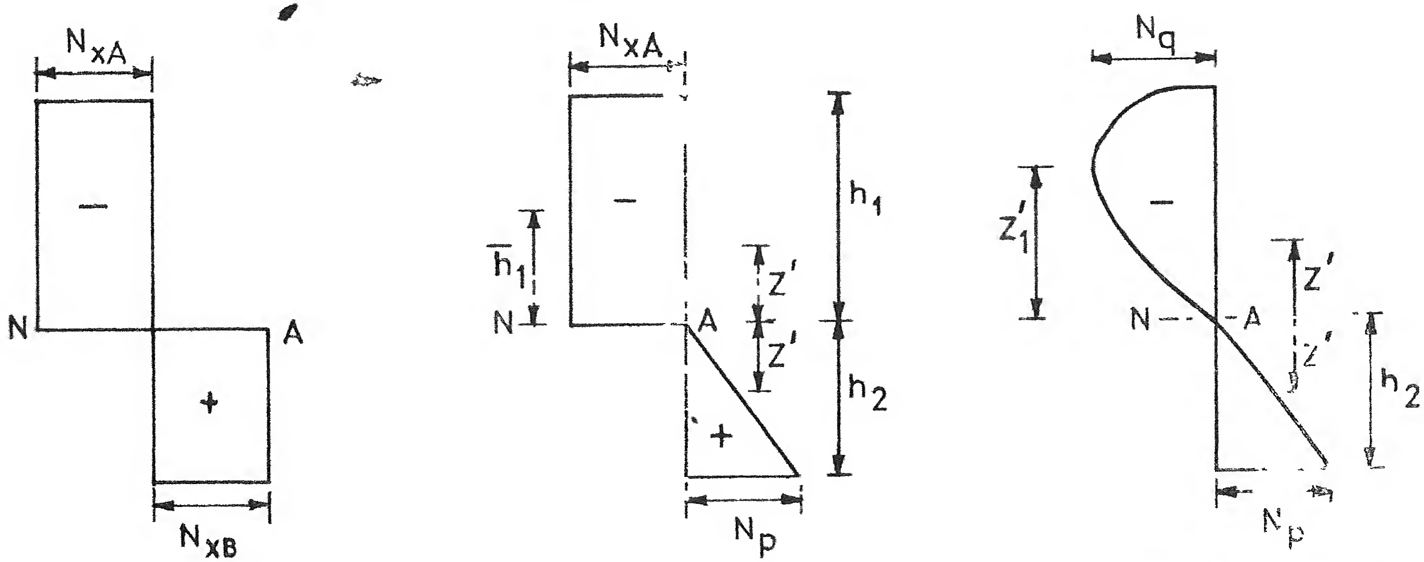
finding out a statically admissible stress field covering the entire domain of the shell.

Any statically admissible stress field has to satisfy the equations of equilibrium (2.15) through (2.18). These four equilibrium equations contain five unknown stress resultants N_x, $N_{x\phi}$, N_{ϕ}, Q_{ϕ} and M_{ϕ}. Hence if one of the unknowns is preassigned then the other can be evaluated by solving the equilibrium equations. With this in view a possible distribution for N_x at collapse is assumed, and the remaining stress resultants are determined. These stress resultants have to satisfy the specified boundary conditions. It is imperative that such solutions are not unique. So it is always possible to improve the load capacity with some other distribution. Three different distributions for N_x are assumed and lower bound solutions are developed for cylindrical shell roofs. Obviously for a valid lower bound the stress resultants should not violate the yield conditions.

Consider the shell shown in Fig. 3.1a. The geometry of the shell and loading are symmetric about the X axis i.e., at $\phi = 0$ and the longitudinal edges are free at $\phi = \phi_0$, where ϕ is the angle measured from the crown to any point on the shell cross section and ϕ_0 is the semicentral angle. The cross section of the shell is divided into two zones,



(a) Shell section



(b) Solution 1 (c) Solution 2 (d) Solution 3

FIG. 3.1 N_x DISTRIBUTION OF VARIOUS LOWER BOUND

viz.,the top compression zone for $0 \leq \phi \leq \beta$ and the bottom tension zone for $\beta \leq \phi \leq \phi_o$, where β is an angle defining the location of the neutral axis. In Fig. 3.1a these values are shown. The stress resultants have different expressions in the two zones but have to satisfy continuity conditions at the region boundary of the two zones.

3.3 Symmetry,Boundary and Continuity Conditions

The stress resultants have to satisfy the following conditions :

(i) Symmetry Conditions :

$$N_{x\phi} = \frac{\partial N_\phi}{\partial \phi} = \frac{\partial M_\phi}{\partial \phi} = 0 \quad \text{at } \phi = 0 \qquad \ldots (3.1)$$

(ii) Continuity Conditions :

The stress resultants $N_{x\phi}$, N_ϕ and M_ϕ and the derivatives $\frac{\partial N_\phi}{\partial \phi}$ and $\frac{\partial M_\phi}{\partial \phi}$ are continuous at $\phi = \beta$.

$$\ldots (3.2)$$

(iii) Boundary Conditions :

$$N_{x\phi} = N_\phi = M_\phi = 0 \quad \text{at } \phi = \phi_o. \qquad \ldots (3.3)$$

The shell is considered as a simply supported at the ends. At the simply supported ends $N_x = 0$ at $x = 0$ and $x = L$.

3.4 Lower Bound Solution 1

In obtaining this solution N_x distribution in the transverse direction is assumed as uniform across the cross section both in tension as well as compression zones (Fig. 3.1b). As N_x is uniform across the section, it is independent of ϕ. This distribution can be expected in long shells if the shell fails as a beam of thin curved cross section. When N_c-yield criterion governs the failure (collapse), then the collapse mode of a long shell is similar to that of a beam.

The magnitude and longitudinal variation of N_x are evaluated from the static equilibrium conditions of the shell, considered as a simply supported beam subjected to uniform gravity loading of intensity p. Then the stress resultants $N_{x\phi}$, N_ϕ, Q_ϕ and M_ϕ are obtained using the equilibrium equations (2.15) through (2.18).

3.4.1 Stress Resultants

Evaluation of N_x:

If N_{XA} and N_{XB} represents the magnitudes of N_x in the compression and tension zones respectively (Fig. 3.1b) then

$$N_x = - N_{XA} \qquad \text{for } 0 \leq \phi \leq \beta \ , \qquad \ldots (3.4a)$$

$$N_x = N_{XB} \qquad \text{for } \beta \leq \phi \leq \phi_0 \ . \qquad \ldots (3.4b)$$

The equilibrium condition that algebraic sum of the forces at any section is equal to zero i.e., $\Sigma N_x \, ds = 0$ and taking compression as negative and tension as positive yields

$$2 N_{XB} R(\phi_o - \beta) - 2 N_{XA} R \beta = 0 ,$$

or,

$$N_{XB} = \frac{N_{XA} \beta}{(\phi_o - \beta)} \qquad \ldots (3.5)$$

Let $\bar{\bar{a}}$ be the lever arm between the tensile and compressive forces and is given by

$$\bar{\bar{a}} = \frac{R(\phi_o \sin \beta - \beta \sin \phi_o)}{\beta(\phi_o - \beta)} .$$

The external bending moment M for simply supported shell subjected to uniform gravity loading at a section a distance x is

$$M = p \, R \, \phi_o(x) \, (L - x) .$$

The moment of resistance M_r at a section a distance x is

$$M_r = 2 \, N_{XA} \, R \, \beta \, \bar{\bar{a}} .$$

Equating the external bending moment M and moment of resistance M_r it is found

$$N_{XA} = \frac{p \, \phi_o}{2aR} (\phi_o - \beta) \, (x) \, (L - x) . \qquad \ldots (3.6a)$$

From Eqs. (3.5) and (3.6a)

$$N_{XB} = \frac{p\ \phi_o}{2aR}\ \beta(x)\ (L - x) \qquad \ldots (3.6b)$$

where

$$a = \phi_o \sin\beta - \beta \sin\phi_o .$$

Eqs. (3.4a), (3.4b), (3.6a) and (3.6b) give

$$N_x = -\frac{p\ \phi_o}{2aR}\ (\phi_o - \beta)\ (x)\ (L - x) \quad \text{for } 0 \leq \phi \leq \beta, \qquad \ldots (3.7)$$

$$N_x = \frac{p\ \phi_o}{2aR}\ \beta(x)\ (L - x) \qquad \text{for } \beta \leq \phi \leq \phi_o \qquad \ldots (3.8)$$

<u>Evaluation of $N_{x\phi}$</u> :

The $N_{x\phi}$ is evaluated in different zones as follows :

The Eq. (2.15) may be rearranged as

$$\frac{\partial N_{x\phi}}{\partial \phi} = -R\,\frac{\partial N_x}{\partial x} . \qquad \ldots (3.9)$$

(i) <u>For $0 \leq \phi \leq \beta$</u>

Differentiating N_x given by Eq. (3.7) with respect to x and substituting in Eq. (3.9), one gets

$$\frac{\partial N_{x\phi}}{\partial \phi} = \frac{p\ \phi_o}{2a}\ (\phi_o - \beta)\ (L - 2x). \qquad \ldots (3.10)$$

Eq. (3.10) gives $N_{x\phi}$ on integration with respect to ϕ as

$$N_{x\phi} = \frac{p\ \phi_o}{2a}\ (\phi_o - \beta)\ (L - 2x)\ \phi + D(x), \quad \ldots (3.11)$$

when D(x) is a function of x only. The value of D(x) is obtained by using the condition $N_{x\phi} = 0$ at $\phi = 0$. Thus the value of D(x) is zero.

(ii) <u>For $\beta \leq \phi \leq \phi_o$</u>

Differentiating N_x given by Eq. (3.8) with respect to x and substituting in Eq. (3.9) yields

$$\frac{\partial N_{x\phi}}{\partial \phi} = - \frac{p\ \phi_o}{2a}\ \beta (L - 2x)\ . \quad \ldots (3.12)$$

On integration with respect to ϕ Eq. (3.12) gives

$$N_{x\phi} = - \frac{p\ \phi_o}{2a}\ \beta (L - 2x)\phi + E(x), \quad \ldots (3.13)$$

where E(x) is a function of x only. E(x) is determined from the condition $N_{x\phi} = 0$ at $\phi = \phi_o$. Thus

$$E(x) = \frac{p\ \phi_o^2}{2a}\ \beta (L - 2x).$$

<u>Evaluation of N_ϕ :</u>

N_ϕ is evaluated in the two zones as follows :

Eq. (2.16) is rewritten as

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = R^2\ \frac{\partial^2 N_x}{\partial x^2} - 2p\ R \cos\ \phi\ . \quad \ldots (3.14)$$

(i) For $0 \leq \phi \leq \beta$

From Eqs. (3.7) and (3.14) it is obtained that

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = \frac{pR}{a} \phi_o(\phi_o - \beta) - 2\, pR \cos\phi \; . \qquad \ldots (3.15)$$

The solution to the differential equation (3.15) is given by

$$N_\phi = A_1 \sin\phi + B_1 \cos\phi - pR\,\phi \sin\phi + \frac{pR}{a} \phi_o(\phi_o - \beta), \qquad \ldots (3.16)$$

where A_1 and B_1 are constants of integration. On differentiation one gets

$$\frac{\partial N_\phi}{\partial \phi} = A_1 \cos\phi - B_1 \sin\phi - pR\,(\sin\phi + \phi\cos\phi)\, . \qquad \ldots (3.17)$$

(ii) For $\beta \leq \phi \leq \phi_o$

Substitution of Eq. (3.8) in Eq. (3.14) yields

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = -\frac{pR}{a} \phi_o \beta - 2p\, R \cos\phi \; . \qquad \ldots (3.18)$$

The solution to the differential equation is

$$N_\phi = A_2 \sin\phi + B_2 \cos\phi - \frac{pR}{a} \phi_o\beta - pR\,\phi \sin\phi \; , \qquad \ldots (3.19)$$

where A_2 and B_2 are constants of integration. On differentiation which yields

$$\frac{\partial N_\phi}{\partial \phi} = A_2 \cos \phi - B_2 \sin \phi - pR(\sin \phi + \phi \cos \phi). \qquad \ldots (3.20)$$

The constants A_1, B_1, A_2 and B_2 are determined from the conditions given by Eqs. (3.1) through (3.3) i.e., (i) $\frac{\partial N_\phi}{\partial \phi} = 0$ at $\phi = 0$, (ii) N_ϕ and $\frac{\partial N_\phi}{\partial \phi}$ are continuous at $\phi = \beta$, (iii) $N_\phi = 0$ at $\phi = \phi_o$. Thus

$$A_1 = 0,$$

$$B_1 = B_{11}\, pR = \frac{pR}{a}\, \phi_o(\beta \cos \phi_o - \phi_o \cos \beta),$$

$$A_2 = A_{22}\, pR = \frac{pR}{a}\, \phi_o^2 \sin \beta,$$

$$B_2 = B_{22}\, pR = \frac{pR}{a}\, \phi_o\, \beta \cos \phi_o.$$

<u>Evaluation of M_ϕ</u> :

M_ϕ is evaluated as follows :

The Eq. (2.17) may be rearranged as

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = -N_\phi R - pR^2 \cos \phi . \qquad \ldots (3.21)$$

(i) <u>For $0 \leq \phi \leq \beta$</u>

From Eqs. (3.16) and (3.21) one gets

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = -B_{11}\, pR^2 \cos \phi + pR^2(\phi \sin \phi - \cos \phi) - \frac{pR^2}{a}\, \phi_o(\phi_o - \beta). \qquad \ldots (3.22)$$

On integration once it is found

$$\frac{\partial M_\phi}{\partial \phi} = -B_{11}\, pR^2 \sin\phi - pR^2 \phi \cos\phi - \frac{pR^2}{a} \phi_o(\phi_o - \beta)\phi + C_1 .$$

... (3 .23)

Second integration yields

$$M_\phi = B_{11}\, pR^2 \cos\phi - pR^2 \left[\frac{\phi^2}{2a} \phi_o(\phi_o - \beta) + \cos\phi + \phi \sin\phi \right] + C_1 \phi + C_2 ,$$

... (3.24)

where C_1 and C_2 are constants of integration.

(ii) For $\beta \leq \phi \leq \phi_o$

Substitution of Eq. (3.19) in Eq. (3.21) yields

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = -A_{22}\, pR^2 \sin\phi - B_{22}\, pR^2 \cos\phi + \frac{pR^2}{a} \phi_o \beta$$

$$+ pR^2 (\phi \sin\phi - \cos\phi) .$$

... (3.25)

First and second integrations of Eq. (3.25) give

$$\frac{\partial M_\phi}{\partial \phi} = A_{22}\, pR^2 \cos\phi - B_{22}\, pR^2 \sin\phi + \frac{pR^2}{a} \phi_o \beta \phi$$

$$- pR^2 \phi \cos\phi + C_3 ,$$

... (3.26)

and

$$M_\phi = A_{22}\, pR^2 \sin\phi + B_{22}\, pR^2 \cos\phi + \frac{pR^2}{2a} \phi_o \beta \phi^2$$

$$- pR^2 (\cos\phi + \phi \sin\phi) + C_3 \phi + C_4 ,$$

... (3.27)

respectively. C_3 and C_4 are constants of integration.

The constants C_1, C_2, C_3 and C_4 are obtained using the conditions given by Eqs. (3.1) through (3.3) i.e., (i) $\frac{\partial M_\phi}{\partial \phi} = 0$ at $\phi = 0$, (ii) M_ϕ and $\frac{\partial M_\phi}{\partial \phi}$ are continuous at $\phi = \beta$, (iii) $M_\phi = 0$ at $\phi = \phi_o$. Thus

$$C_1 = 0 \ ,$$

$$C_2 = C_{22}\, pR^2 = \frac{pR^2}{a}\left[\phi_o(\phi_o - \beta)\left(1 + \frac{\phi_o \beta}{2}\right) + a \cos \phi_o\right],$$

$$C_3 = -C_{33}\, pR^2 = -\frac{pR^2}{a}\phi_o^2 \beta ,$$

$$C_4 = C_{44}\, pR^2 = \frac{pR^2}{a}\left[a \cos \phi_o - \phi_o \beta\left(1 - \frac{\phi_o^2}{2}\right)\right].$$

Evaluation of Q_ϕ :

Q_ϕ is evaluated as follows :

Eq. (2.18) is rewritten as

$$Q_\phi = \frac{1}{R}\frac{\partial M_\phi}{\partial \phi} \ . \qquad \ldots (3.28)$$

From Eqs. (3.23), (3.26) and (3.28), it is found

$$Q_\phi = -B_{11}\, pR \sin \phi - pR\left[\frac{\phi_o}{a}(\phi_o - \beta)\phi + \phi \cos \phi\right] \quad \text{for } 0 \leq \phi \leq \beta \qquad \ldots (3.29)$$

and

$$Q_\phi = A_{22}\, pR \cos \phi - B_{22}\, pR \sin \phi + pR\left(\frac{\phi_o \beta \phi}{a} - \phi \cos \phi\right) - C_{33}\, pR$$

$$\text{for } \beta \leq \phi \leq \phi_o . \qquad \ldots (3.30)$$

The stress resultants after substitution of the constants of integration boil down to :

(i) For $0 \leq \phi \leq \beta$

$$N_x = -\frac{p\,\phi_o}{2aR}\,(\phi_o - \beta)\,(x)\,(L - x) \qquad \ldots (3.31)$$

$$N_{x\phi} = \frac{p\,\phi_o}{2a}\,(\phi_o - \beta)\,(\phi)\,(L - 2x) \qquad \ldots (3.32)$$

$$N_\phi = -\frac{pR}{a}\Big[a\,\phi\,\sin\phi - \phi_o(\beta\,\cos\phi_o - \phi_o\,\cos\beta)\cos\phi - \phi_o(\phi_o - \beta)\Big] \qquad \ldots (3.33)$$

$$M_\phi = -\frac{pR^2}{a}\Big[a\,(\cos\phi + \phi\,\sin\phi - \cos\phi_o) - \phi_o(\beta\,\cos\phi_o - \phi_o\,\cos\beta)\,\cos\phi + \frac{\phi_o}{2}\,(\phi_o - \beta)(\phi^2 - \phi_o\beta - 2)\Big] \qquad \ldots (3.34)$$

$$Q_\phi = -\frac{pR}{a}\Big[a\,\phi\,\cos\phi + \phi_o(\beta\,\cos\phi_o - \phi_o\,\cos\beta)\,\sin\phi + \phi_o(\phi_o - \beta)\,\phi\Big] \qquad \ldots (3.35)$$

(ii) For $\beta \leq \phi \leq \phi_o$

$$N_x = \frac{p\,\phi_o}{2aR}\,\beta(x)\,(L - x) \qquad \ldots (3.36)$$

$$N_{x\phi} = \frac{p\,\phi_o}{2a}\,\beta\,(\phi_o - \phi)\,(L - 2x) \qquad \ldots (3.37)$$

$$N_\phi = -\frac{pR}{a}\Big[a\,\phi\,\sin\phi - \phi_o^2\,\sin\beta\,\sin\phi + \phi_o\,\beta\,(1 - \cos\phi_o\,\cos\phi)\Big] \qquad \ldots (3.38)$$

$$M_\phi = -\frac{pR^2}{a}\Big[a(\cos\phi + \phi\sin\phi - \cos\phi_o) - \phi_o^2 \sin\beta \sin\phi$$

$$- \phi_o\beta \cos\phi_o \cos\phi - \frac{\phi_o\beta}{2}((\phi_o-\phi)^2 - 2)\Big] \quad \ldots (3.39)$$

$$Q_\phi = -\frac{pR}{a}\Big[a\phi\cos\phi - \phi_o^2 \sin\beta \cos\phi + \phi_o\beta\cos\phi_o \sin\phi$$

$$+ \phi_o\beta(\phi_o - \phi)\Big] \quad \ldots (3.40)$$

The expressions for the stress resultants N_ϕ, M_ϕ and Q_ϕ indicate that they do not vary with x.

3.4.2 Determination of Limit Load

The limit load is computed from the two independent yield conditions and the minimum of the two values is taken as the collapse load. Let p_{L1} and p_{L2} represent the two lower bounds to the collapse load found from N_c and M_ϕ–N_ϕ yield criteria respectively. They are evaluated as follows :

(i) Due to N_c-yield criterion : From Eq. (3.31) it can easily be found that N_x is maximum at the centre of span. Thus

$$N_{x\,max} = -\frac{p\,\phi_o}{8aR}(\phi_o - \beta)L^2 . \quad \ldots (3.41)$$

Yield condition (2.26) gives

$$p_{L1} = \frac{8aR\,N_c}{\phi_o(\phi_o - \beta)L^2} . \quad \ldots (3.42)$$

Then p_1, the limit load in nondimensional form is obtained after dividing by σ_c as

$$p_1 = \frac{p_{L1}}{\sigma_c} = \frac{8a}{\phi_o(\phi_o - \beta)(\frac{L}{R})^2 \ (\frac{R}{t})} \ . \qquad \ldots (3.43)$$

(ii) Due to M_ϕ-N_ϕ yield criterion : The failure of a cylindrical shell occurs in the transverse direction where M_ϕ is maximum. In a simply supported cylindrical shell with free longitudinal edges, M_ϕ is maximum at its crown. Both M_ϕ and N_ϕ do not vary in sign for the entire shell and N_ϕ is always compressive. Since N_ϕ and M_ϕ are independent of x, it is understood that if once yielding is initiated at a point then the entire shell yields along the crown. Let $N_{\phi c}$ and $M_{\phi c}$ represent the values of M_ϕ and N_ϕ at crown. Then Eqs. (3.33) and (3.34) yield

$$N_{\phi c} = \frac{pR}{a} \left[\phi_o \ (\beta \ \cos \phi_o - \phi_o \ \cos \beta) + \phi_o(\phi_o - \beta) \right], \qquad \ldots (3.44)$$

$$M_{\phi c} = - \frac{pR^2}{a} \left[a(1 - \cos \phi_o) - \phi_o(\beta \ \cos \phi_o - \phi_o \ \cos \beta) - \frac{\phi_o}{2} \ (\phi_o - \beta) \ (\phi_o \beta + 2) \right] . \qquad \ldots (3.45)$$

Let p_2 denotes the nondimensional load. p_2 is found from the yield condition given by Eq. (2.30) using the Eqs. (3.44) and (3.45). Thus

$$p_2 = \frac{p_{L2}}{\sigma_c} = \frac{ba}{D_1} , \qquad \ldots (3.46)$$

where

$$D_1 = 4\left(\frac{R}{t}\right)^2 \left[a\ (1-\cos\ \phi_o) - \phi_o(\beta\ \cos\ \phi_o - \phi_o\ \cos\beta) - \frac{\phi_o}{2}(\phi_o - \beta)(\phi_o\beta + 2) \right] + c\left(\frac{R}{t}\right) \left[\phi_o\ (\beta\ \cos\ \phi_o - \phi_o\ \cos\beta) + \phi_o(\phi_o - \beta) \right] .$$

Failure of the shell occurs either in the longitudinal direction due to N_c-yield criterion or in the transverse direction due to N_ϕ-N_ϕ yield criterion whichever is critical. Denoting the lower bound in nondimensional form as p_o which is equal to the minimum of the two values p_1 and p_2.

3.5 Lower Bound Solution 2

In the solution 1, N_x is taken as uniform across the cross section, which requires uniform distribution for steel in longitudinal direction in the tension zone at collapse. But the elastic analysis reveals that N_x distribution is almost linear in the tension zone at working loads. Therefore, the assumed uniform distribution for N_x may not be satisfactory. In view of this the distribution for N_x is modified in the solution 2 as shown in the Fig. 3.1c. In this, the N_x is uniform across the cross section in the compression zone and it varies linearly with depth z' from the neutral

axis in the tension zone. The stress resultants are evaluated on similar lines.

3.5.1 Stress Resultants :

Evaluation of N_x :

Let $\bar{h}_1$ be the vertical height of the centre of total compression from the neutral axis (Fig. 3.1c). It is same as the centroid of the circular arc of the shell above the neutral axis. Thus

$$\bar{h}_1 = R\left(\frac{\sin\beta}{\beta} - \cos\beta\right).$$

Let h_1 and h_2 be the depths of compression and tension zones (Fig. 3.1a) respectively which are given by

$$h_1 = R(1 - \cos\beta),$$

$$h_2 = R(\cos\beta - \cos\phi_0).$$

If z' is the vertical distance from the neutral axis to any point across the section which is given as

$$z' = R(1 - \cos\phi) \quad \text{for } 0 \leq \phi \leq \beta,$$

$$z' = R(\cos\phi - \cos\beta) \text{ for } \beta \leq \phi \leq \phi_0.$$

The maximum value of N_x in the tension zone and the uniform value of N_x in the compression zone are denoted by N_p and N_{XA} respectively (Fig. 3.1c). Then the variation of N_x in the two zones is expressed as

$$N_x = -N_{XA} \qquad \text{for } 0 \leq \phi \leq \beta, \qquad \ldots (3.47a)$$

$$N_x = \frac{N_p\, z'}{h_2} = \frac{N_p(\cos\beta - \cos\phi)}{(\cos\beta - \cos\phi_o)} \qquad \text{for } \beta \leq \phi \leq \phi_o. \qquad \ldots (3.47b)$$

The static equilibrium condition i.e., $\Sigma N_x \, ds = 0$ at any section x yields

$$2\int_o^{\beta} N_x\, R\, d\phi + 2\int_{\beta}^{\phi_o} N_x\, R\, d\phi = 0 \qquad \ldots (3.48)$$

Substituting N_x from Eqs. (3.47a) and (3.47b) and integrating, the following is obtained

$$-2RN_{XA}\,\beta + 2RN_p \frac{[(\phi_o - \beta)\cos\beta - \sin\phi_o + \sin\beta]}{(\cos\beta - \cos\phi_o)} = 0,$$

which yields

$$N_p = \frac{N_{XA}\,\beta(\cos\beta - \cos\phi_o)}{[(\phi_o - \beta)\cos\beta - \sin\phi_o + \sin\beta]}. \qquad \ldots (3.49)$$

Taking moments about the neutral axis, the moment of resistance M_r at a section a distance x is given by

$$M_r = 2\int_o^{\beta} N_x\, R\, d\phi\, z' + 2\int_o^{\phi_o} N_x\, R\, d\phi\, z'. \qquad \ldots (3.50)$$

Substituting N_x from Eqs. (3.47a) and (3.47b) and performing integration, M_r is evaluated as

$$M_r = 2\, N_{XA}\,\beta\, R^2\, a_1,$$

where

$$a_1 = \frac{1}{a}\left[\frac{(\phi_o - \beta)}{2}\left(1 + \frac{\sin 2\beta}{\beta}\right) + \frac{\sin 2\phi_o}{4} + \frac{\sin 2\beta}{4} + \frac{\sin \beta}{\beta}(\sin \beta - \sin \phi_o) - \cos \beta \sin \phi_o\right]$$

$$a = (\phi_o - \beta) \cos \beta - \sin \phi_o + \sin \beta .$$

The external bending moment M at a section a distance x, under uniform gravity loading p is

$$M = pR\,\phi_o(x)\,(L - x).$$

Equating the external bending moment M and moment of resistance M_r one gets

$$N_{XA} = \frac{p\,\phi_o(x)\,(L - x)}{2\,a_1\,\beta\,R}, \qquad \ldots (3.51)$$

Using Eqs. (3.47b),(3.49) and (3.51), N_x in the tension zone is found. Thus

$$N_x = \frac{p\,\phi_o}{2aa_1R}(\cos \beta - \cos \phi)\,(x)\,(L - x). \qquad \ldots (3.52)$$

Substitution of Eqs. (3.51) and (3.52) in (3.47a) and (3.47b) respectively yield

$$N_x = -\frac{p\,\phi_o(x)\,(L - x)}{2a_1\,\beta\,R} \qquad \text{for } 0 \leq \phi \leq \beta, \ldots (3.53)$$

$$N_x = \frac{p\,\phi_o}{2a_1aR}(\cos \beta - \cos \phi)(x)(L - x) \quad \text{for } \beta \leq \phi \leq \phi_o .$$

$$\ldots (3.54)$$

Evaluation of $N_{x\phi}$:

$N_{x\phi}$ is evaluated as follows :

Eq. (2.15) gives

$$\frac{\partial N_{x\phi}}{\partial \phi} = - R \frac{\partial N_x}{\partial x} . \qquad \ldots (3.55)$$

(i) For $0 \leq \phi \leq \beta$

Eqs. (3.53) and (3.55) yield

$$\frac{\partial N_{x\phi}}{\partial \phi} = \frac{p \phi_o}{2a_1 \beta} (L - 2x) . \qquad \ldots (3.56)$$

On integration with respect to ϕ it is found

$$N_{x\phi} = \frac{p \phi_o \phi}{2a_1 \beta} (L - 2x) + D(x), \qquad \ldots (3.57)$$

where $D(x)$ is a function of x only. $D(x)$ is found by using the condition $N_{x\phi} = 0$ at $\phi = 0$. Thus the value of $D(x)$ is zero.

(ii) For $\beta \leq \phi \leq \phi_o$

Eqs. (3.54) and (3.55) give

$$\frac{\partial N_{x\phi}}{\partial \phi} = - \frac{p \phi_o}{2aa_1} (\cos\beta - \cos\phi) (L - 2x) \ldots (3.58)$$

On integration with respect to ϕ one gets

$$N_{x\phi} = - \frac{p \phi_o}{2aa_1} (\phi \cos\beta - \sin\phi)(L-2x)+E(x), \qquad \ldots (3.59)$$

where $E(x)$ is a function of x only. $E(x)$ is determined from the condition $N_{x\phi} = 0$ at $\phi = \phi_o$. Thus

$$E(x) = \frac{p\,\phi_o}{2aa_1}\,(\phi_o \cos\beta - \sin\phi_o)(L-2x). \qquad \ldots (3.60)$$

<u>Evaluation of N_ϕ</u> :

N_ϕ is evaluated as follows :

Eq. (2.16) is rewritten as

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = R^2 \frac{\partial^2 N_x}{\partial x^2} - 2pR\cos\phi\,. \qquad \ldots (3.61)$$

(i) <u>For $0 \le \phi \le \beta$</u>

Substitution of Eq. (3.53) in Eq. (3.61) yields

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = \frac{pR\,\phi_o}{a_1\beta} - 2pR\cos\phi\,. \qquad \ldots (3.62)$$

The differential equation has a solution of the form

$$N_\phi = A_1 \sin\phi + B_1 \cos\phi + \frac{pR\,\phi_o}{a_1\beta} - pR\,\phi\sin\phi\,, \qquad \ldots (3.63)$$

where A_1 and B_1 are constants of integration. On differentiation one gets

$$\frac{\partial N_\phi}{\partial \phi} = A_1\cos\phi - B_1\sin\phi - pR(\sin\phi + \phi\cos\phi). \qquad \ldots (3.64)$$

(ii) <u>For $\beta \le \phi \le \phi_o$</u>

Eqs. (3.54) and (3.61) give

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = - \frac{pR\ \phi_o}{aa_1}(\cos\beta - \cos\phi) - 2pR\cos\phi\ . \qquad \ldots (3.65)$$

The solution of the differential equation is

$$N_\phi + A_2 \sin\phi + B_2 \cos\phi - pR\left[\frac{\phi_o \cos\beta}{aa_1} - \frac{\phi_o}{2aa_1}\phi\sin\phi + \phi\sin\phi\right], \qquad \ldots (3.66)$$

where A_2 and B_2 are constants of integration. On differentiation Eq. (3.66) gives

$$\frac{\partial N_\phi}{\partial\phi} = A_2\cos\phi - B_2\sin\phi - pR(\sin\phi + \phi\cos\phi)(1 - \frac{\phi_o}{2aa_1})\ . \qquad \ldots (3.67)$$

The constants A_1, B_1, A_2 and B_2 are determined from the conditions (3.1) through (3.3) i.e., (i) $\frac{\partial N}{\partial\phi} = 0$ at $\phi = 0$, (ii) N_ϕ and $\frac{\partial N_\phi}{\partial\phi}$ are continuous at $\phi = \beta$, (iii) $N_\phi = 0$ at $\phi = \phi_o$. The constants are rewritten as $A_1 = A_{11}\, pR$, $B_1 = B_{11}\, pR$, $A_2 = A_{22}\, pR$ and $B_2 = B_{22}\, pR$, where

$$A_{11} = 0\ ,$$

$$B_{11} = \phi_o \tan\phi_o + \frac{\phi_o}{4aa_1}\left[\tan\phi_o\left(\frac{4\cos\beta}{\sin\phi_o} + 2\beta - 2\phi_o - \sin 2\beta\right) - \cos 2\beta - 3\right] - \frac{\phi_o}{a_1\beta}(\sin\beta\tan\phi_o + \cos\beta),$$

$$A_{22} = \frac{\phi_o}{4aa_1}(\sin 2\beta - 2\beta) + \frac{\phi_o}{a_1}\frac{\sin\beta}{\beta},$$

$$B_{22} = \phi_o \tan\phi_o \left[1 - \frac{\sin\beta}{a_1\beta} + \frac{1}{4aa_1}\left(\frac{4\cos\beta}{\sin\phi_o} + 2\beta - 2\phi_o - \sin 2\beta\right)\right].$$

Evaluation of M_ϕ :

M_ϕ is determined as given below :

Eq. (2.17) may be rewritten as

$$\frac{\partial^2 M_\phi}{\partial\phi^2} = - N_\phi R - pR^2 \cos\phi . \qquad \ldots (3.68)$$

(1) For $0 \leq \phi \leq \beta$

Substitution of Eq. (3.63) in Eq. (3.68) yields

$$\frac{\partial^2 M_\phi}{\partial\phi^2} = -B_{11}\, pR^2 \cos\phi - pR^2\left(\frac{\phi_o}{a_1\beta} - \phi\sin\phi\right) - pR^2 \cos\phi . \qquad \ldots (3.69)$$

On first integration it is found that

$$\frac{\partial M_\phi}{\partial\phi} = -B_{11}\, pR^2 \sin\phi - pR^2\left(\frac{\phi_o\phi}{a_1\beta} + \phi\cos\phi\right) + C_1 . \qquad \ldots (3.70)$$

Second integration gives

$$M_\phi = B_{11}\, pR^2 \cos\phi - pR^2\left(\frac{\phi_o\phi^2}{2a_1\beta} + \cos\phi + \phi\sin\phi\right) + C_1\phi + C_2, \qquad \ldots (3.71)$$

where C_1 and C_2 are constants of integration.

$$C_{11} = 0,$$

$$C_{33} = \frac{\phi_o}{aa_1}(\sin\beta - \beta\cos\beta) - \frac{\phi_o}{aa_1},$$

$$C_{44} = \cos\phi_o + \frac{\phi_o^2}{a_1} - \frac{\phi_o}{aa_1}\Big[\cos\phi_o + (1 + \frac{\phi_o^2}{2})\cos\beta + \phi_o(\sin\beta - \beta\cos\beta)\Big]$$

$$C_{22} = \frac{\phi_o}{2aa_1}(4 + \beta^2)\cos\beta + \frac{\phi_o}{2a_1\beta}(2+\beta^2) + C_{33}\beta + C_{44}.$$

<u>Evaluation of Q_ϕ</u> :

Q_ϕ is determined as follows :

Eq. (2.18) is rewritten as

$$Q_\phi = \frac{1}{R}\frac{\partial M_\phi}{\partial \phi}. \qquad \text{... (3.75)}$$

From Eqs. (3.70),(3.73) and (3.75), it is found

$$Q_\phi = - B_{11}\,pR\sin\phi - pR(\frac{\phi_o\phi}{a_1\beta} + \phi\cos\phi)$$

$$\text{for } 0 \leq \phi \leq \beta \qquad \text{... (3.76)}$$

and

$$Q_\phi = A_{22}\,pR\cos\phi - B_{22}\,pR\sin\phi + pR\Big[\frac{\phi_o\cos\beta}{aa_1}\phi - \frac{\phi_o}{2aa_1}(\sin\phi - \phi\cos\phi) - \phi\cos\phi\Big] + C_{33}\,pR$$

$$\text{for } \beta \leq \phi \leq \phi_o \qquad \text{... (3.77)}$$

The stress resultants are summarised below :

(i) For $0 \leq \phi \leq \beta$

$$N_x = -\frac{p\ \phi_o}{2a_1 \beta R}\ (x)\ (L - x) \qquad \ldots (3.78)$$

$$N_{x\phi} = \frac{p\ \phi_o}{2\beta\ a_1}\ \phi (L - 2x) \qquad \ldots (3.79)$$

$$N_\phi = -pR \left[-B_{11} \cos \phi - \frac{\phi_o}{\beta\ a_1} + \phi \sin \phi \right] \qquad \ldots (3.80)$$

$$M_\phi = -pR^2 \left[\cos \phi - B_{11} \cos \phi + \phi \sin \phi + \frac{\phi_o\ \phi^2}{2\beta\ a_1} - C_{22} \right]$$
$$\ldots (3.81)$$

$$Q_\phi = -pR \left[B_{11} \sin \phi + \phi \cos \phi + \frac{\phi_o \phi}{\beta\ a_1} \right] \qquad \ldots (3.82)$$

(ii) For $\beta \leq \phi \leq \phi_o$

$$N_x = \frac{p\ \phi_o}{2aa_1 R}\ (\cos \beta - \cos \phi)(x)(L - x) \qquad \ldots (3.83)$$

$$N_{x\phi} = \frac{p\ \phi_o}{2aa_1} \left[(\phi_o - \phi) \cos \beta + \sin \phi - \sin \phi_o \right] (L - 2x)$$
$$\ldots (3.84)$$

$$N_\phi = -pR \left[\phi \sin \phi + \frac{\phi_o}{aa_1} \left(\cos \beta - \frac{\phi \sin \phi}{2} \right) - A_{22} \sin \phi \right.$$
$$\left. - B_{22} \cos \phi \right] \qquad \ldots (3.85)$$

$$M_\phi = -pR^2 \left[\cos \phi + \phi \sin \phi - \frac{\phi_o}{2aa_1} (2 \cos \phi + \phi \sin \phi \right.$$
$$\left. + \phi^2 \cos \beta) - A_{22} \sin \phi - B_{22} \cos \phi - C_{33\phi} - C_{44} \right]$$
$$\ldots (3.86)$$

$$Q_\phi = -pR \left[\phi \cos\phi - \frac{\phi_o}{2aa_1} (\phi \cos \phi - \sin \phi + 2\phi \cos\beta) - \right.$$

$$\left. - A_{22} \cos\phi + B_{22} \sin\phi - C_{33} \right] \qquad \dots (3.87)$$

3.5.2 Determination of Limit Load

The limit load is determined by using the yield conditions. Let p_{L1} and p_{L2} represent the two lower bounds found from N_c and M_ϕ-N_ϕ yield criteria respectively. They are evaluated as follows :

(i) Due to N_c-yield criterion : The maximum value of N_x in compression is determined from Eq. (3.78) at $x = \frac{L}{2}$. Thus

$$N_{x\ max} = -\frac{p\ \phi_o L^2}{8a_1\beta\ R} . \qquad \dots (3.88)$$

The yield condition (2.26) gives

$$p_{L1} = \frac{8a_1\ \beta R\ N_c}{\phi_o\ L^2} . \qquad \dots (3.89)$$

Denoting the nondimensional load as p_1, it is given as

$$p_1 = \frac{p_{L1}}{\sigma_c} = \frac{8a_1\ \beta}{\phi_o (\frac{L}{R})^2 (\frac{R}{t})} . \qquad (3.90)$$

(ii) Due to M_ϕ-N_ϕ yield criterion : Failure of the shell occurs at the corwn in the transverse direction (see section 3.4.2). Let $N_{\phi c}$ and $M_{\phi c}$ represent the values of N_ϕ and M_ϕ at crown i.e., at $\phi = 0$. Eqs. (3.80) and (3.81) yield

$$N_{\phi c} = -\, pR(-B_{11} - \frac{\phi_o}{\beta a_1}) \,, \qquad \ldots (3.91)$$

$$M_{\phi c} = -pR^2(1 - B_{11} - C_{22}). \qquad \ldots (3.92)$$

Let p_2 represents the load in nondimensional form. Substituting $N_{\phi c}$ and $M_{\phi c}$ in the yield condition given by Eq. (2.30) it is found

$$p_2 = \frac{p_{L2}}{\sigma_c} = \frac{b}{\left[4(\frac{R}{t})^2(1-C_{22}-B_{11}) + c(\frac{R}{t})(B_{11} + \frac{\phi_o}{\beta a_1}) \right]} \,. \qquad \ldots (3.93)$$

The minimum of the two values p_1 and p_2 is denoted by p_o, which is the actual lower bound.

3.6 Lower Bound Solution 3

In the lower bound solutions 1 and 2, N_x is taken as uniform in the compression zone. This uniform distribution for N_x is possible only if the shell fails as a beam of curved cross section. Depending on the geometric parameters of the shell, it is possible the failure of the shell in the transverse direction due to M_ϕ-N_ϕ yield criterion may precede the N_c-yield criterion. If this failure occurs, then the distribution of N_x may not be uniform across the section in the compression zone. Logically the distribution of N_x in the compression zone is assumed as parabolic, varying with vertical distance from the neutral axis. In the parabolic

variation, the maximum value of N_x in the transverse direction occurs at an angle β_1 (Fig. 3.1a). This angle β_1 is kept as variable and its value is determined for a shell of given geometry such that it gives optimal value for the ultimate load. The distribution for N_x in the tension zone is taken as linear variation as in the lower bound solution 2. This is shown in Fig. 3.1d.

3.6.1 Stress Resultants

Evaluation of N_x :

N_x is determined as described below :

N_q and N_p represents the maximum values of N_x in the compression and tension zones respectively as shown in the Fig. 3.1d. As described earlier the variation of N_x in the transverse direction is given by

$$N_x = -\frac{1}{z_1'^2} N_q z' (2z_1' - z') \quad \text{for } 0 \leq \phi \leq \beta, \quad \ldots (3.94)$$

and

$$N_x = \frac{N_p z'}{h_2} \quad \text{for } \beta \leq \phi \leq \phi_0, \quad \ldots (3.95)$$

where z_1' is the vertical distance from neutral axis defining the position of the maximum compression and h_2 is depth of the tension zone. Thus

$$z' = R (\cos \phi - \cos \beta) \quad \text{for } 0 \leq \phi \leq \beta,$$

$$z' = R(\cos\beta - \cos\phi) \qquad \text{for } \beta \le \phi \le \phi_o,$$

$$z_1' = R(\cos\beta_1 - \cos\beta),$$

$$h_2 = R(\cos\beta - \cos\phi_o).$$

These are indicated in the Figs. 3.1a and 3.1d. On substitution of the above in Eqs. (3.94) and (3.95) it is obtained

$$N_x = -\frac{N_q}{(\cos\beta_1 - \cos\beta)^2}(2\cos\beta_1 - \cos\beta - \cos\phi)(\cos\phi - \cos\beta) \qquad \text{for } 0 \le \phi \le \beta \qquad \ldots (3.96)$$

and

$$N_x = \frac{N_p(\cos\beta - \cos\phi)}{(\cos\beta - \cos\phi_o)} \qquad \text{for } \beta \le \phi \le \phi_o. \qquad \ldots (3.97)$$

The static equilibrium condition i.e., $\Sigma N_x \, ds = 0$ at any section of the shell yields

$$2\int_0^\beta N_x R \, d\phi + 2\int_\beta^{\phi_o} N_x R d\phi = 0.$$

Substituting Eqs. (3.96) and (3.97) and on carrying out the integration it is found

$$-\frac{N_q R}{(\cos\beta_1 - \cos\beta)^2}\left[2\cos\beta_1(\sin\beta - \beta\cos\beta) + \frac{\beta}{2}\cos 2\beta - \frac{\sin 2\beta}{4}\right] + \frac{N_p R}{(\cos\beta - \cos\phi_o)}\left[(\phi_o - \beta)\cos\beta - \sin\phi_o + \sin\beta\right] = 0,$$

which gives

$$N_p = \frac{N_q (\cos\beta - \cos\phi_o)}{a_1(\cos\beta_1 - \cos\beta)^2} , \qquad \ldots (3.98)$$

where

$$a_1 = \frac{\left[(\phi_o - \beta) \cos\beta - \sin\phi_o + \sin\beta \right]}{\left[2\cos\beta_1(\sin\beta - \beta\cos\beta) + \frac{\beta}{2}\cos 2\beta - \frac{\sin 2\beta}{4} \right]} .$$

The moment of resistance M_r at any section is found by taking moments about the neutral axis. Thus

$$M_r = 2\int_0^{\beta} N_x R \, d\phi \, z' + 2\int_{\beta}^{\phi_o} N_x R \, d\phi \, z' .$$

Substituting Eqs. (3.96) and (3.97) and performing integration one gets

$$M_r = \frac{2N_q R^2 a_2}{(\cos\beta_1 - \cos\beta)^2} , \qquad \ldots (3.99)$$

where

$$a_2 = \beta \cos\beta_1 (2 + \cos 2\beta) + \sin 2\beta \left(\frac{7}{12}\cos\beta - \frac{3}{2}\cos\beta_1\right)$$

$$- \frac{2}{3}\sin\beta - \frac{\beta}{2}\cos\beta \cos 2\beta + \frac{1}{4a_1}\left[(\phi_o - \beta)(4 + 2\cos 2\beta) \right.$$

$$\left. + \sin 2\phi_o - 8\cos\beta \sin\phi_o + 3\sin 2\beta \right] .$$

The external bending moment M at any section x, subjected to uniform gravity loading is

$$M = pR\,\phi_o(x)\,(L - x). \qquad (3.100)$$

Equating Eqs. (3.99) and (3.100) one gets

$$N_q = \frac{p\,\phi_o}{2a_2R}\,(\cos\beta_1 - \cos\beta)^2\,(x)\,(L - x). \qquad \ldots (3.101)$$

Using Eqs. (3.96), (3.97), (3.98) and (3.101), N_x can be expressed as

$$N_x = -\frac{p\,\phi_o}{2a_2R}\,(2\cos\beta_1 - \cos\beta - \cos\phi)\,(\cos\phi - \cos\beta)(x)(L-x)$$

$$\text{for } 0 \leq \phi \leq \beta \qquad \ldots (3.102)$$

and

$$N_x = \frac{p\,\phi_o\,(\cos\beta - \cos\phi)}{2a_1a_2R}\,(x)(L-x) \quad \text{for } \beta \leq \phi \leq \phi_o \ldots (3.103)$$

<u>Evaluation of $N_{x\phi}$</u> :

$N_{x\phi}$ is determined as described below :

Eq. (3.15) is rewritten as

$$\frac{\partial N_{x\phi}}{\partial\phi} = -R\,\frac{\partial N_x}{\partial x}. \qquad (3.104)$$

(i) <u>For $0 \leq \phi \leq \beta$</u>

Substitution of Eq. (3.102) in Eq. (3.104) yields

$$\frac{\partial N_{x\phi}}{\partial\phi} = \frac{p\,\phi_o}{2a_2}\,(2\cos\beta_1 - \cos\beta - \cos\phi)(\cos\phi - \cos\beta)(L-2x).$$

On integrating with respect to ϕ

$$N_{x\phi} = \frac{p\ \phi_o}{2a_2}\left[2\cos\beta_1(\sin\phi - \phi\cos\beta) + \frac{\phi}{2}\cos 2\beta - \frac{\sin 2\phi}{4}\right](L - 2x) + D(x), \quad \ldots (3.105)$$

where $D(x)$ is a function of x only.

(ii) For $\beta \leq \phi \leq \phi_o$

Eqs. (3.103) and (3.104) give

$$\frac{\partial N_{x\phi}}{\partial\phi} = -\frac{p\ \phi_o}{2a_1 a_2}(\cos\beta - \cos\phi)(L - 2x),$$

which on integrating with respect to ϕ yields

$$N_{x\varphi} = -\frac{p\ \phi_o}{2a_1 a_2}(\phi\cos\beta - \sin\phi)(L - 2x) + E(x), \quad \ldots (3.106)$$

where $E(x)$ is a function of x only.

Evaluation of N_ϕ :

N_ϕ is evaluated as follows :

Eq. (2.16) results

$$N_\phi + \frac{\partial^2 N_\phi}{\partial\phi^2} = R^2\frac{\partial^2 N_x}{\partial x^2} - 2\ pR\cos\phi\ . \quad \ldots (3.107)$$

(i) For $0 \leq \phi \leq \beta$

From the Eqs. (3.102) and (3.107) it is obtained

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = \frac{pR\ \phi_o}{a_2} (2 \cos \beta_1 - \cos \beta - \cos \phi)(\cos \phi - \cos \beta)$$

$$- 2pR \cos \phi \ . \qquad \ldots (3.108)$$

The differential equation (3.108) has the solution in the form

$$N_\phi = A_1 \sin \phi + B_1 \cos \phi + \frac{pR}{a_2} \phi_o (\phi \sin \phi \cos \beta_1$$

$$- 2 \cos \beta \cos \beta_1 + \frac{\cos 2\beta}{2} + \frac{\cos 2\phi}{6}) - pR \phi \sin \phi ,$$

$$\ldots (3.109)$$

where A_1 and B_1 are constants of integration. On differentiation the following is found

$$\frac{\partial N_\phi}{\partial \phi} = A_1 \cos \phi - B_1 \sin \phi + \frac{pR}{a_2} \phi_o \left[\cos \beta_1 (\sin \phi + \phi \cos \phi) \right.$$

$$\left. - \frac{\sin 2\phi}{3} \right] - pR (\sin \phi + \phi \cos \phi). \qquad \ldots (3.110)$$

(ii) <u>For $\beta \leq \phi \leq \phi_o$</u>

Substitution of Eq. (3.103) in Eq. (3.107) gives

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = - \frac{pR\ \phi_o}{a_1 a_2} (\cos \beta - \cos \phi) - 2pR \cos \phi .$$

$$\ldots (3.111)$$

The solution to the differential equation is

$$N_\phi = A_2 \sin \phi + B_2 \cos \phi - \frac{pR}{a_1 a_2} \phi_o \cos \beta + \frac{pR}{2a_1 a_2} \phi_o \phi \sin \phi$$

$$- pR \phi \sin \phi , \qquad \ldots (3.112)$$

where A_2 and B_2 are constants of integration. On differentiation one gets

$$\frac{\partial N_\phi}{\partial \phi} = A_2 \cos \phi - B_2 \sin \phi + \frac{pR}{2a_1 a_2} \phi_o(\sin \phi + \phi \cos \phi)$$

$$- pR(\sin \phi + \phi \cos \phi). \qquad \ldots (3.113)$$

The constants are modified as $A_1 = A_{11}\, pR$, $B_1 = B_{11}\, pR$, $A_2 = A_{22}\, pR$, $B_2 = B_{22}\, pR$.

Evaluation of M_ϕ :

M_ϕ is determined as follows :

Eq. (3.17) yields

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = -N_\phi R - pR^2 \cos \phi . \qquad \ldots (3.114)$$

(i) For $0 \leq \phi \leq \beta$

From Eqs. (3.109) and (3.114) it is obtained

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = -A_{11}\, pR^2 \sin \phi - B_{11}\, pR^2 \cos \phi - \frac{pR^2}{a_2} \phi_o(\cos \beta_1\, \phi \sin \phi$$

$$- 2 \cos \beta_1 \cos \beta + \frac{\cos 2\beta}{2} + \frac{\cos 2\phi}{6}) + pR^2(\phi \sin \phi - \cos \phi).$$

$$\ldots (3.115)$$

First and second integrations with respect ϕ give

$$\frac{\partial M_\phi}{\partial \phi} = A_{11}\, pR^2 \cos\phi - B_{11}\, pR^2 \sin\phi - \frac{pR^2}{2}\phi_o \Big[\cos\beta_1 (\sin\phi - \phi\cos\phi) - 2\phi\cos\beta_1 \cos\beta + \frac{\cos 2\beta}{2}\phi + \frac{\sin 2\phi}{12}\Big] - pR^2\phi \cos\phi + C_1 \qquad \ldots (3.116)$$

and

$$M_\phi = A_{11}\, pR^2 \sin\phi + B_{11}\, pR^2 \cos\phi - \frac{pR^2}{a_2}\phi_o \Big[\frac{\cos 2\beta}{4}\phi^2 - \cos\beta_1 (2\cos\phi + \phi\sin\phi) - (\cos\beta_1 \cos\beta)\phi^2 - \frac{\cos 2\phi}{24}\Big] - pR^2(\cos\phi + \phi\sin\phi) + C_1\phi + C_2 \qquad \ldots (3.117)$$

where C_1 and C_2 are constants of integration.

(ii) For $\beta \leq \phi \leq \phi_o$

Substitution of Eq. (3.112) in Eq. (3.114) yields

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = -A_{22}\, pR^2 \sin\phi - B_{22}\, pR^2 \cos\phi + pR^2 \Big(\frac{\phi_o \cos\beta}{a_1 a_2} - \frac{\phi_o}{2a_1 a_2}\phi\sin\phi + \phi\sin\phi - \cos\phi\Big). \qquad \ldots (3.118)$$

On first integration it is found

$$\frac{\partial M_\phi}{\partial \phi} = A_{22}\, pR^2 \cos\phi - B_{22}\, pR^2 \sin\phi + pR^2 \Big[\frac{\phi_o \cos\beta}{a_1 a_2}\phi - \frac{\phi_o}{2a_1 a_2}(\sin\phi - \phi\cos\phi) - \phi\cos\phi\Big] + C_3 \qquad \ldots (3.119)$$

where C_3 is a constant of integration. By second integration one gets

$$M_\phi = A_{22}\, pR^2 \sin\phi + B_{22}\, pR^2 \cos\phi + pR^2 \left[\frac{\phi_o}{2a_1 a_2} (2\cos\phi + \phi\sin\phi + \phi^2 \cos\beta) - \cos\phi - \phi\sin\phi \right] + C_3\phi + C_4, \quad \ldots (3.120)$$

where C_4 is a constant of integration. The constants are modified as $C_1 = C_{11}\, pR^2$, $C_2 = C_{22}\, pR^2$, $C_3 = C_{33}\, pR^2$ and $C_4 = C_{44}\, pR^2$.

Evaluation of Q_ϕ :

Q_ϕ is determined as described below :

Eq. (2.18) yields

$$Q_\phi = \frac{1}{R} \frac{\partial M_\phi}{\partial\phi}. \quad (3.121)$$

From the Eqs. (3.116), (3.119) and (3.121) it is found

$$Q_\phi = A_{11}\, pR\cos\phi - B_{11}\, pR\sin\phi - \frac{pR}{a_2}\phi_o \left[\cos\beta_1 (\sin\phi - \phi\cos\phi) - 2\phi\cos\beta_1 \cos\beta + \frac{\cos 2\beta}{2}\phi + \frac{\sin 2\phi}{12} \right]$$
$$- pR\phi\cos\phi + C_{11}\, pR \quad \text{for } 0 \le \phi \le \beta \quad \ldots (3.122)$$

and

$$Q_\phi = A_{22}\, pR \cos\phi - B_{22}\, pR \sin\phi + pR \left[\frac{\phi_o \cos\beta}{a_1 a_2} \phi - \frac{1}{2a_1 a_2} (\sin\phi - \phi \cos\phi) - \phi \cos\phi \right] + C_{33}\, pR$$

$$\text{for } \beta \leq \phi \leq \phi_o. \qquad \ldots (3.123)$$

The values of $D(x)$, $E(x)$, A_{11}, B_{11}, A_{22}, B_{22}, C_{11}, C_{22}, C_{33}, C_{44} are obtained applying the conditions (3.1) through (3.3). They are given by

$$D(x) = 0,$$

$$E(x) = \frac{p\,\phi_o}{2a_1 a_2} (\phi_o \cos\beta - \sin\phi_o)(L - 2x),$$

$$A_{11} = 0,$$

$$B_{11} = \phi_o \tan\phi_o + \frac{\phi_o}{4a_1 a_2} \left[f_3 \tan\phi_o - f_2 \cot\beta - 2(1+\beta\cot\beta) \right] + \frac{\phi_o}{a_2} \left[\cos\beta_1 (1+\beta\cot\beta) - \frac{2}{3}\cos\beta - f_1 \tan\phi_o - \frac{f_1}{6}\cot\beta \right],$$

$$A_{22} = \frac{\phi_o}{6a_2} f_1 + \frac{\phi_o}{4a_1 a_2} f_2,$$

$$B_{22} = \phi_o \tan\phi_o + \frac{\phi_o}{4a_1 a_2} f_3 \tan\phi_o - \frac{\phi_o}{a_2} f_1 \tan\phi_o,$$

$$C_{11} = 0$$

$$C_{22} = \cos\phi_o - \frac{\phi_o}{2a_1 a_2} f_6 - \frac{\phi_o}{4a_2} \left[(1 - \phi_o) f_4 + f_7 \right],$$

$$C_{33} = \frac{\phi_o}{a_1 a_2} f_5 - \frac{\phi_o}{4a_2} f_4 ,$$

$$C_{44} = \cos \phi_o - \frac{\phi_o}{2a_1 a_2} f_8 + \frac{\phi_o^2}{4a_2} f_4 ,$$

where the constants f_i, $i = 1,8$ are given by

$$f_1 = 6\beta \cos \beta_1 - 3 \cos \beta_1 \sin 2\beta - 2 \sin \beta (1 - \cos 2\beta),$$

$$f_2 = \sin 2\beta - 2\beta ,$$

$$f_3 = 4 \cos \beta \operatorname{cosec} \phi_o - \sin 2\beta - 2(\phi_o - \beta),$$

$$f_4 = 2\beta \cos 2\beta + 8 \cos \beta_1 (\sin \beta - \beta \cos \beta) - \sin 2\beta ,$$

$$f_5 = \sin \beta - \beta \cos \beta ,$$

$$f_6 = 2(\phi_o - \beta) \sin \beta + 2 \cos \phi_o + (\phi_o - \beta)^2 \cos \beta - 2 \cos \beta ,$$

$$f_7 = 4(4 + \beta^2) \cos \beta_1 \cos \beta - (2.5 + \beta^2) \cos \beta ,$$

$$f_8 = 2 \cos \phi_o (2 + \phi_o^2 - 2 \phi_o \beta) \cos \beta + 2 \phi_o \sin \beta .$$

The stress resultants are summarised as follows :

For $0 \leq \phi \leq \beta$

$$N_x = - \frac{p \phi_o}{2a_2 R} (2 \cos \beta_1 - \cos \beta - \cos \phi) (\cos \phi - \cos \beta)(x)(L-x) \quad \ldots (3.124)$$

$$N_{x\phi} = p \frac{\phi_o}{a_2} \left[\cos \beta_1 (\sin \phi - \phi \cos \beta) + \frac{\phi}{4} \cos 2\beta - \frac{\sin 2\phi}{8} \right] (L-2x) \quad \ldots (3.125)$$

$$N_\phi = pR\left[B_{11}\cos\phi - \phi\sin\phi + \frac{\phi_o}{a_2}(\phi\cos\beta_1 \quad \sin\phi - \right.$$

$$\left. - 2\cos\beta\ \cos\beta_1 + \frac{\cos 2\beta}{2} + \frac{\cos 2\phi}{6})\right] \qquad \ldots (3.126)$$

$$M_\phi = pR^2\ (B_{11}\cos\phi + C_{22} - \cos\phi - \phi\sin\phi) +$$

$$+ pR^2\ \frac{\phi_o}{a_2}\left[\cos\beta_1(2\cos\phi + \phi\sin\phi) + \right.$$

$$\left. + \phi^2(\cos\beta_1\ \cos\beta - \frac{\cos 2\beta}{4}) + \frac{\cos 2\phi}{24}\right] \qquad \ldots (3.127)$$

$$Q_\phi = -pR\ \frac{\phi_o}{a_2}\left[\cos\beta_1(\sin\phi - \phi\cos\phi) - 2\phi\cos\beta_1\ \cos\beta\right.$$

$$\left. + \frac{\cos 2\beta}{2}\ \phi + \frac{\sin 2\phi}{12}\right] - pR(B_{11}\sin\phi + \phi\cos\phi - C_{11})$$

$$\ldots (3.128)$$

<u>For $\beta \leq \phi \leq \phi_o$</u>

$$N_x = p\ \frac{\phi_o}{2a_1 a_2 R}\ (\cos\beta - \cos\phi)\ (x)\ (L - 2x) \qquad \ldots (3.129)$$

$$N_{x\phi} = p\ \frac{\phi_o}{2a_1 a_2}\left[\sin\phi - \sin\phi_o + (\phi_o - \phi)\cos\beta\right](L - 2x)$$

$$\ldots (3.130)$$

$$N_\phi = pR\left[A_{22}\sin\phi + B_{22}\cos\phi - \phi\sin\phi + \right.$$

$$\left. + \frac{\phi_o}{2a_1 a_2}(\phi\sin\phi - 2\cos\beta)\right] \qquad \ldots (3.131)$$

$$M_\phi = pR^2 \Big[A_{22} \sin\phi + B_{22} \cos\phi + C_{33}\phi + C_{44} - \cos\phi - \phi \sin\phi$$

$$+ \frac{\phi_o}{2a_1 a_2} (2\cos\phi + \phi\sin\phi + \phi^2 \cos\beta) \Big] \qquad \ldots (3.132)$$

$$Q_\phi = pR \Big[A_{22} \cos\phi - B_{22} \sin\phi + C_{33} - \phi\cos\phi$$

$$+ \frac{\phi_o \cos\beta}{a_1 a_2} \phi - \frac{(\sin\phi - \phi\cos\phi)}{2a_1 a_2} \Big] \qquad \ldots (3.133)$$

3.6.2 Determination of Limit Load

Limit load is found out from the yield conditions. Let p_{L1} and p_{L2} be the lower bounds obtained from the N_c and the M_ϕ-N_ϕ yield criteria respectively. They are evaluated as follows :

(i) Due to N_c-yield criterion : The maximum value of N_x will occur at the centre of the span of the shell in the longitudinal direction and at $\phi = \beta_1$ in the transverse direction. Substituting $x = \frac{L}{2}$ and $\phi = \beta_1$ in Eq. (3.124) it is found that

$$N_{x\,max} = \frac{-p\,\phi_o}{8a_2 R} (\cos\beta_1 - \cos\beta)^2 L^2 . \qquad \ldots (3.134)$$

The yield condition (2.26) gives

$$p_{L1} = \frac{8a_2\, R\, N_c}{\phi_o\, L^2\, (\cos\beta_1 - \cos\beta)^2} . \qquad \ldots (3.135)$$

The nondimensional load p_1 is evaluated as

$$p_1 = \frac{p_{L1}}{\sigma_c} = \frac{8a_2}{\phi_o(\frac{L}{R})^2 (\frac{R}{t}) (\cos\beta_1 - \cos\beta)^2} \quad . \qquad \ldots (3.136)$$

(ii) Due to N_ϕ-M_ϕ yield criterion : Failure of the shell occurs at the crown in the transverse direction (see section 3.4.2). Let $N_{\phi c}$ and $M_{\phi c}$ represent the values of N_ϕ and M_ϕ at crown i.e., at $\phi = 0$. The Eqs. (3.126) and (3.127) yield

$$N_{\phi c} = - pR \left[-B_{11} - \frac{\phi_o}{a_2} (\frac{1}{6} + \frac{\cos 2\beta}{2} - 2 \cos\beta \cos\beta_1) \right] ,$$

$$\ldots (3.137)$$

$$M_{\phi c} = - pR^2 \left[1 - C_{22} - B_{11} - \frac{\phi_o}{a_2} (\frac{1}{24} + 2 \cos\beta_1) \right] .$$

$$\ldots (3.138)$$

p_2 denotes the load in nondimensional form. Substitution of $N_{\phi c}$ and $M_{\phi c}$ in the yield condition given by Eq. (2.50) yield

$$p_2 = \frac{p_{L2}}{\sigma_c} = \frac{b}{D_1} , \qquad \ldots (3.139)$$

where

$$D_1 = 4(\frac{R}{t})^2 (1 - C_{22} - B_{11} - \frac{\phi_o}{24 a_2} - 2 \frac{\phi_o}{a_2} \cos\beta_1)$$
$$+ c(\frac{R}{t}) \left[B_{11} + \frac{\phi_o}{a_2} (\frac{1}{6} + \frac{\cos 2\beta}{2} - 2 \cos\beta \cos\beta_1) \right] .$$

Then p_o is the minimum of the two values p_1 and p_2.

3.6.3 Optimal Value of β_1

In the Eqs. (3.136) and (3.139), β_1 is a variable while other parameters are constant. The value of β_1 is determined in such a way that p_1 and p_2 are optimal. The values of β_1 are determined numerically. For different values of ϕ_o, values of β_1 are found for a given value of β. There values are tabulated in Table 3.1.

Table 3.1 - Values of β_1 in solution 3.

ϕ_o	β	β_1 (in radians)
30^o	$\frac{2}{3}\phi_o$	0.1854
	$\frac{\phi_o}{2}$	0.1091
35^o	$\frac{2}{3}\phi_o$	0.2327
	$\frac{\phi_o}{2}$	0.1309
40^o	$\frac{2}{3}\phi_o$	0.3345
	$\frac{\phi_o}{2}$	0.1527
45^o	$\frac{2}{3}\phi_o$	0.3818
	$\frac{\phi_o}{2}$	0.1636

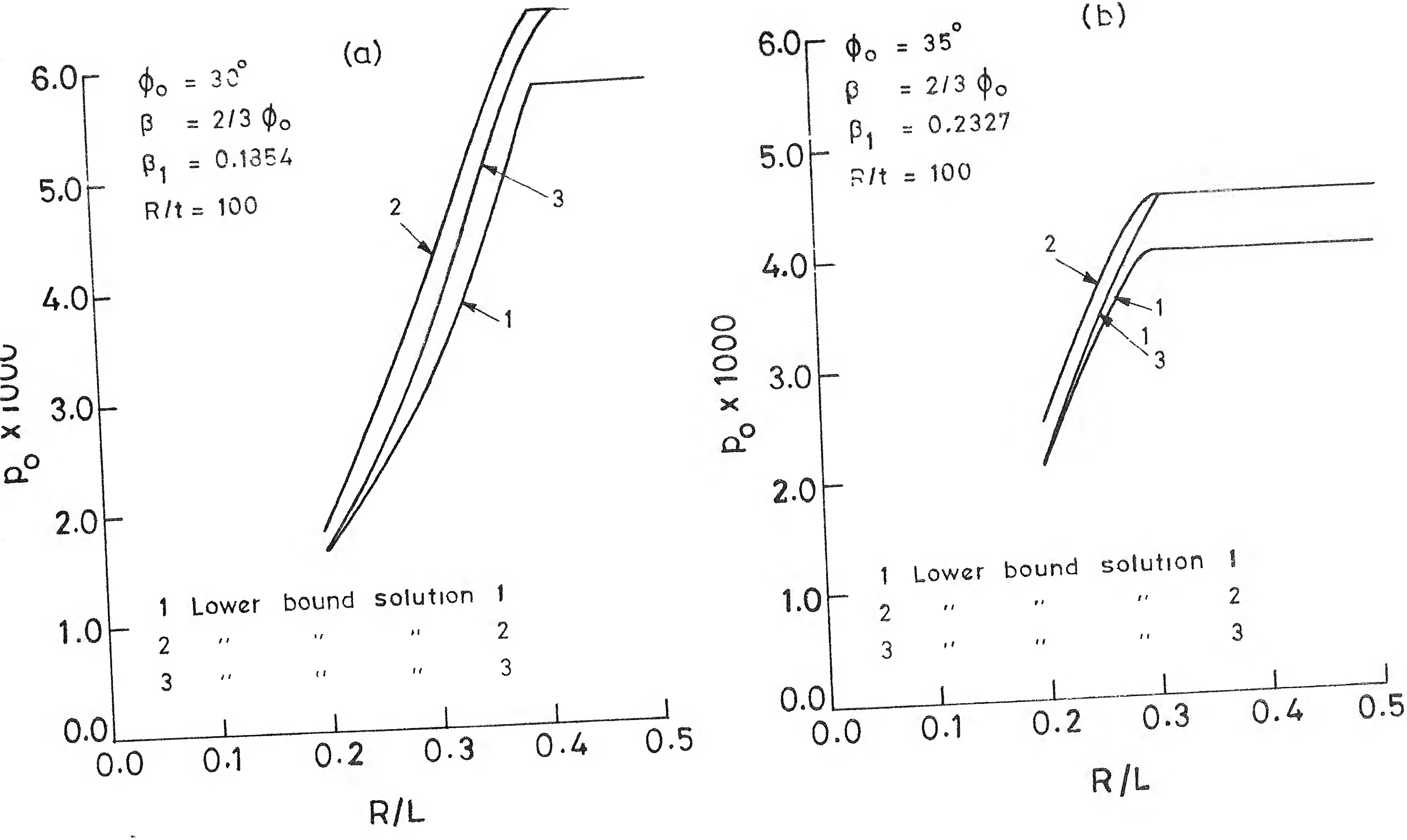


FIG. 3.2 VARIATION OF p_o WITH R/L

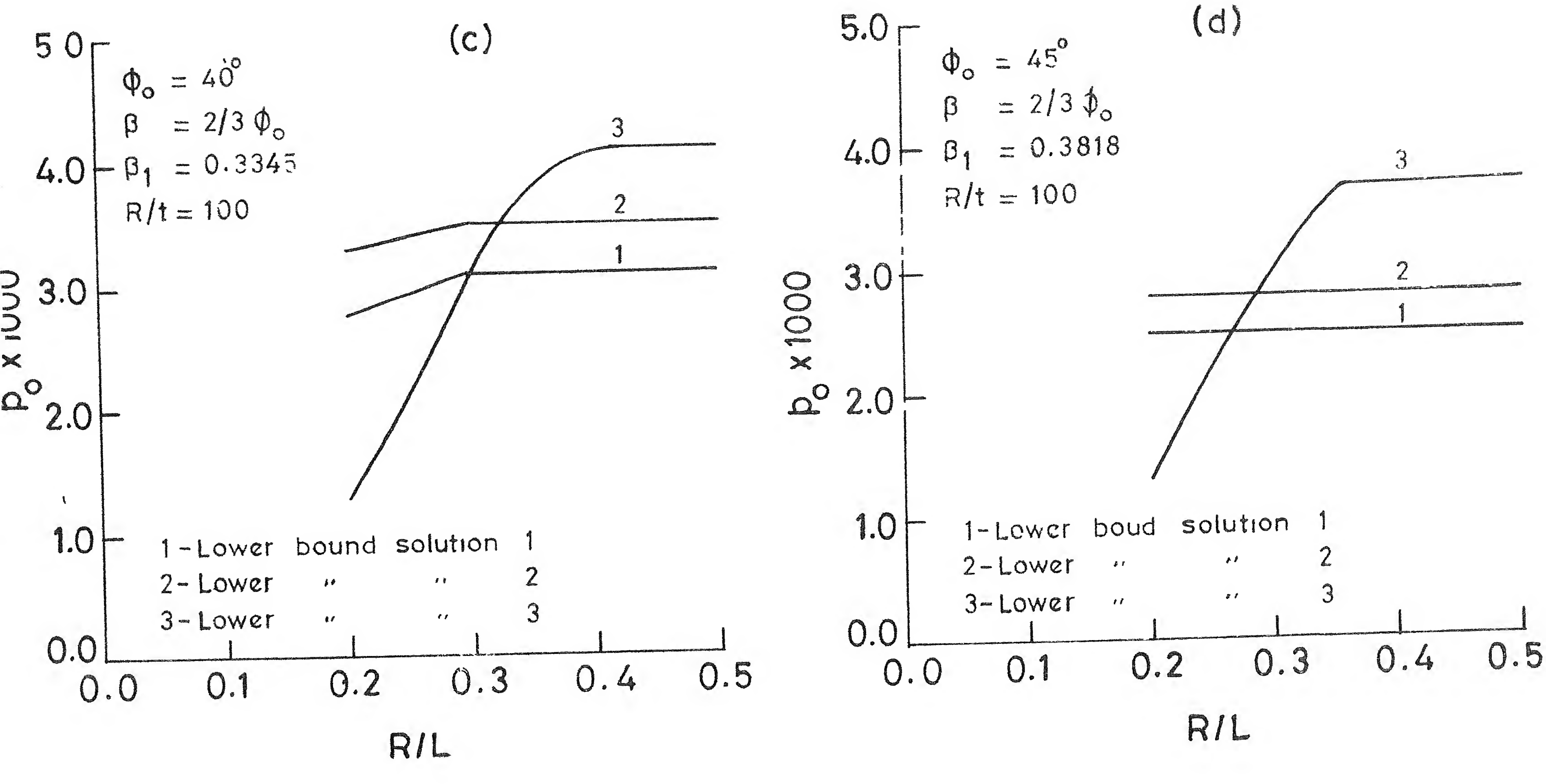


FIG 3.2 VARIATION OF p_o WITH R/L

3.7 Variation of p_o

Shells with different geometric parameters are analysed by the proposed lower bound solutions. The values of p_o are found for different $\frac{R}{L}$ ratios with a fixed $\frac{R}{t}$ ratio equal to 100 and for various values of ϕ_o. The angle β which defines the position of the neutral axis is taken as equal $\frac{2}{3}\ \phi_o$. The value of β_1 in the lower bound solution 3 is taken from the Table 3.1 The variation of p_o and with $\frac{R}{L}$ ratio for various values ϕ_o is plotted (Fig. 3.2). These graphs are useful to estimate the ultimate load for any given shell.

3.8 Analysis of Shells

Two shells with different geometric parameters are analysed by both elastic (26) and the proposed lower bound solutions. Elastic analysis is carried out for a working load of intensity 290 kg/m^2. The ultimate load and the stress resultants are calculated at collapse. In the limit analysis the amount of transverse reinforcement is taken such that $\mu = 0.236$. σ_c and σ_{sy} are equal to 150 kg/cm^2 and 2600 kg/cm^2 respectively.

For estimating the collapse load for a given thickness of shell, 't' will be the total thickness of the shell while using N_c-yield criterion and will be the effective thickness while using M_ϕ-N_ϕ yield criterion.

3.8.1 Numerical Example 1 (Long shell) :

The geometric parameters of the shell 1 are :

$\phi_o = 30^\circ$, $\beta = \frac{2}{3}\,\phi_o$, $\frac{L}{R} = 3.333$, R = 8.0 m and t = 8.0 cms.

This shell fails due to N_c-yield criterion. Solution 2 gives the best lower bound and is equal to 640 kg/m^2. Hence the load factor is equal to 2.2. The values of the stress resultants at critical sections at collapse are tabulated in Table 3.2. The stress resultants in reduced form at critical sections are plotted both for elastic and limit analyses for a comparative study (Fig. 3.3). The stress resultants in reduced form are $\frac{N_x}{N_o}$, $\frac{N_{x\phi}}{S_o}$, $\frac{N_\phi}{N_o}$ and $\frac{M_\phi}{M_o}$ where $S_o = 0.1\,N_o$. The variation of the stress resultants with $\frac{y_1}{h}$ are plotted, where y_1 and h are the vertical distance measured from the springing level and depth of the shell respectively.

3.8.2. Numerical Example 2 (Medium shell) :

The geometric parameters of the shell 2 are :

$\phi_o = 45^\circ$, $\beta = \frac{2}{3}\,\phi_o$, $\beta_1 = 0.3318$, $\frac{L}{R} = 2.5$, R = 8.0 m and t = 8.0 cms.

This shell fails due to M_ϕ-N_ϕ yield criterion. The solution 3 gives the best lower bound which is equal to 560 kg/m^2 and hence the load factor is 1.95. The values of

Table C.2 – Reduced stress resultants at critical sections for shell 1 (Long Shell)

ϕ	$\frac{N_x}{N_0}$ $(x = L/2)$	$\frac{N_\phi}{N_0}$	$\frac{M_\phi}{M_0}$	$\frac{N_{x\phi}}{S_0}$ $(x = 0)$
0°	-1.0000	-0.0697	-0.7072	0.0000
5°	-1.0000	-0.0670	-0.6658	1.0476
10°	-1.0000	-0.0589	-0.5493	2.0952
15°	-1.0000	-0.0456	-0.3808	3.1428
20°	0.0026	-0.0271	-0.1985	4.1889
25°	1.9374	-0.0078	-0.0548	3.2082
30°	4.266[illegible]	0.0000	0.0000	0.0000

Table 3.3 - Reduced stress resultants at critical sections for shell 2 (Medium Shell)

ϕ	$\frac{N_x}{N_o}$ $(x = L/2)$	$\frac{N_\phi}{N_o}$	$\frac{M_\phi}{M_o}$	$\frac{N_{x\phi}}{S_o}$ $(x = 0)$
0°	0.3187	-0.0457	-0.9376	0.0000
5°	0.1465	-0.0469	-0.9340	-0.2936
10°	-0.1108	-0.0496	-0.8900	-0.3329
15°	-0.4340	-0.0514	-0.8094	0.0477
20°	-0.6089	-0.0492	-0.6849	0.8393
25°	-0.6079	-0.0408	-0.5164	1.7793
30°	0.0007	-0.0272	-0.3304	2.2825
35°	0.3418	-0.0131	-0.1596	2.0487
40°	0.7283	-0.0028	-0.0416	1.3064
45°	1.3000	0.0000	0.0000	0.0000

Table 3.4 - Lower bound solutions

Shell Type	Lower bound solution 1 (p in kg/m^2)		Lower bound solution 2 (p in kg/m^2)		Lower bound solution 3 (p in kg/m^2)		Best lower bound (p in kg/m^2)
	N_c Criterion	$M_\phi - N_\phi$ Criterion	N_c Criterion	$M_\phi - N_\phi$ Criterion	N_c Criterion	$M_\phi - N_\phi$ Criterion	
Long Shell	538.0	854.0	640.0	956.0	540.0	960.0	640.0
Medium Shell	2098.0	320.0	1382.0	420.0	812.0	560.0	560.0

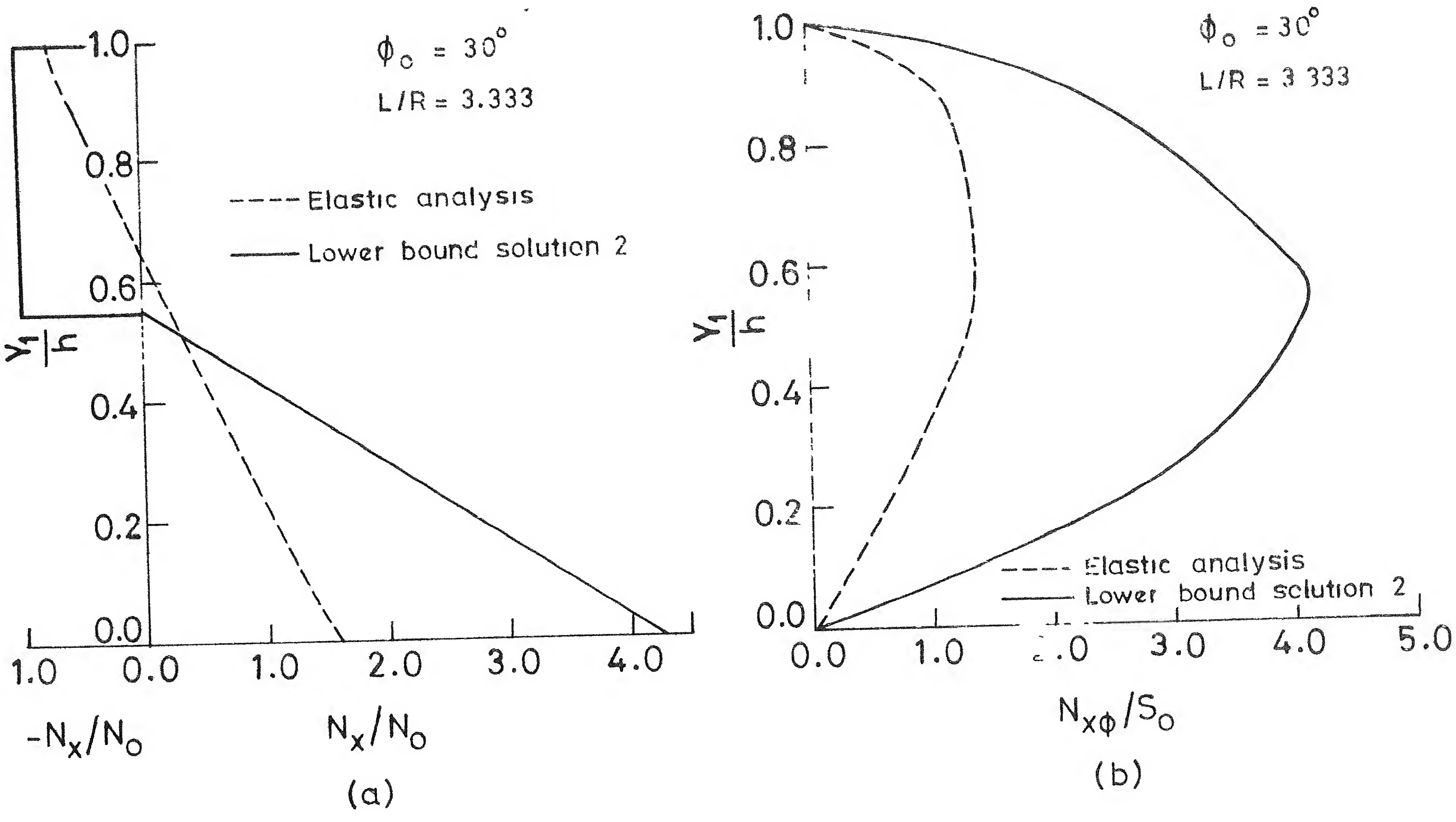


FIG.3.3 DISTRIBUTION OF STRESS RESULTANTS - EXAMPLE 1

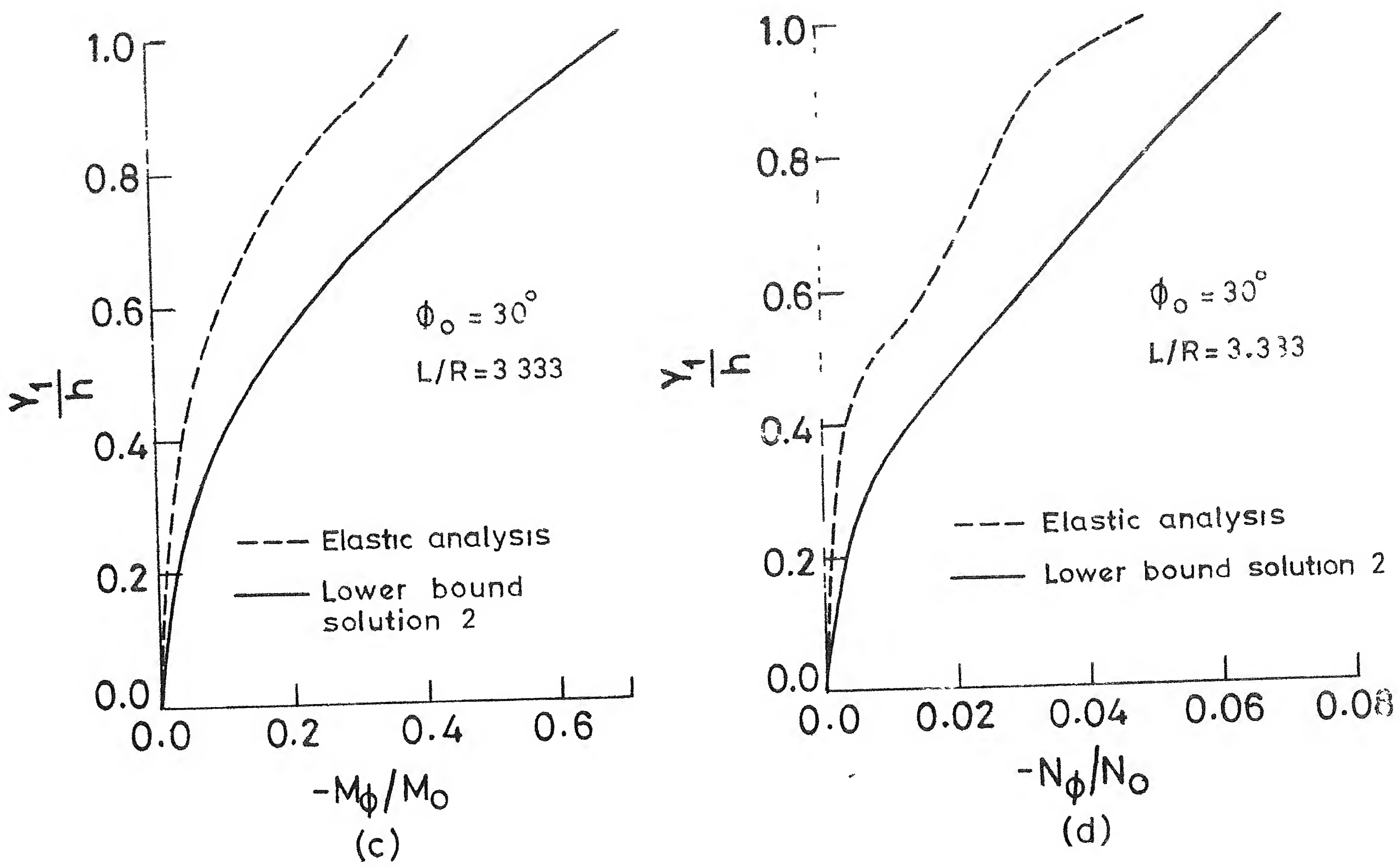


FIG.3.3 DISTRIBUTION OF STRESS RESULTANTS-EXAMPLE 1

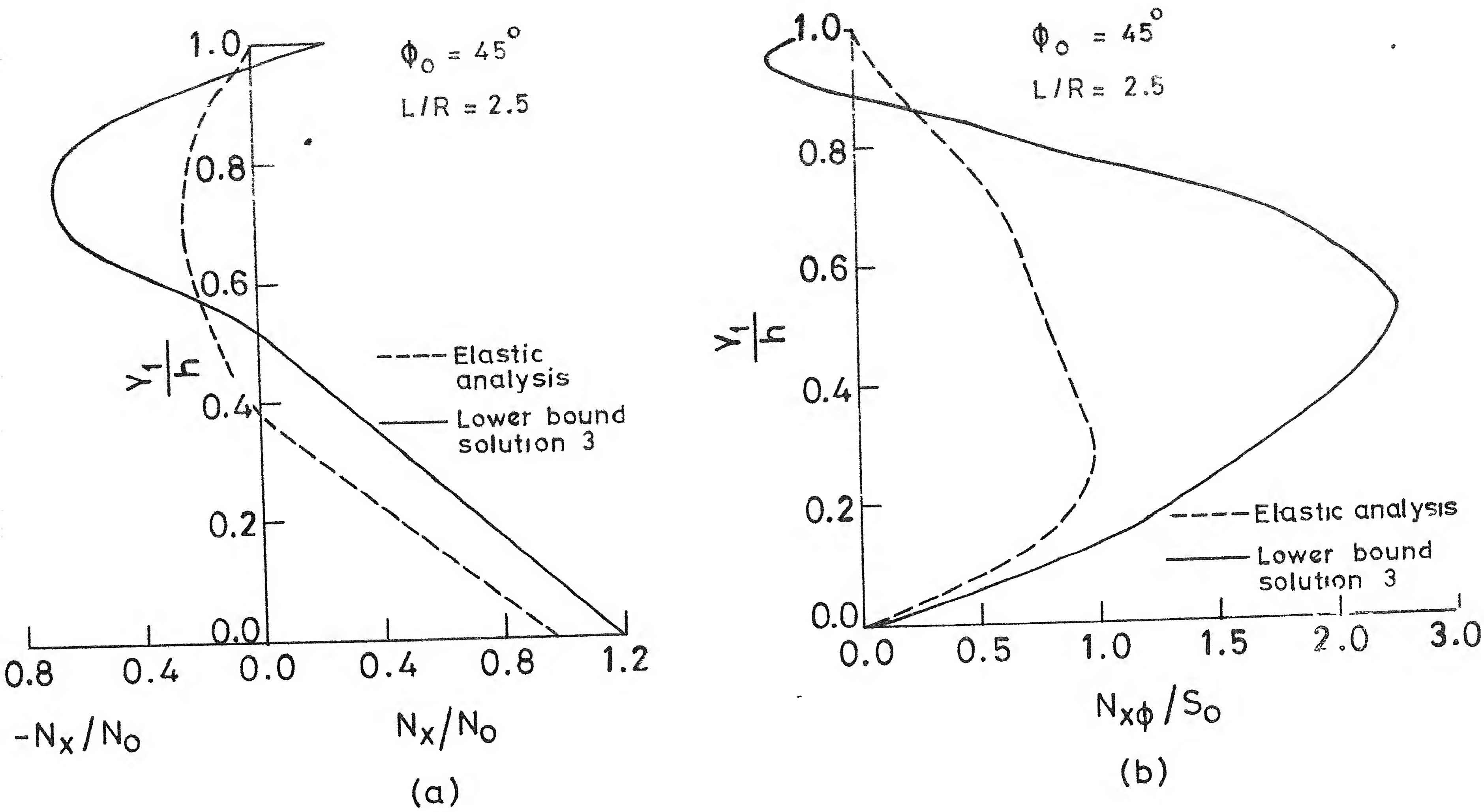


FIG. 3.4 DISTRIBUTION OF STRESS RESULTANTS-EXAMPLE 2

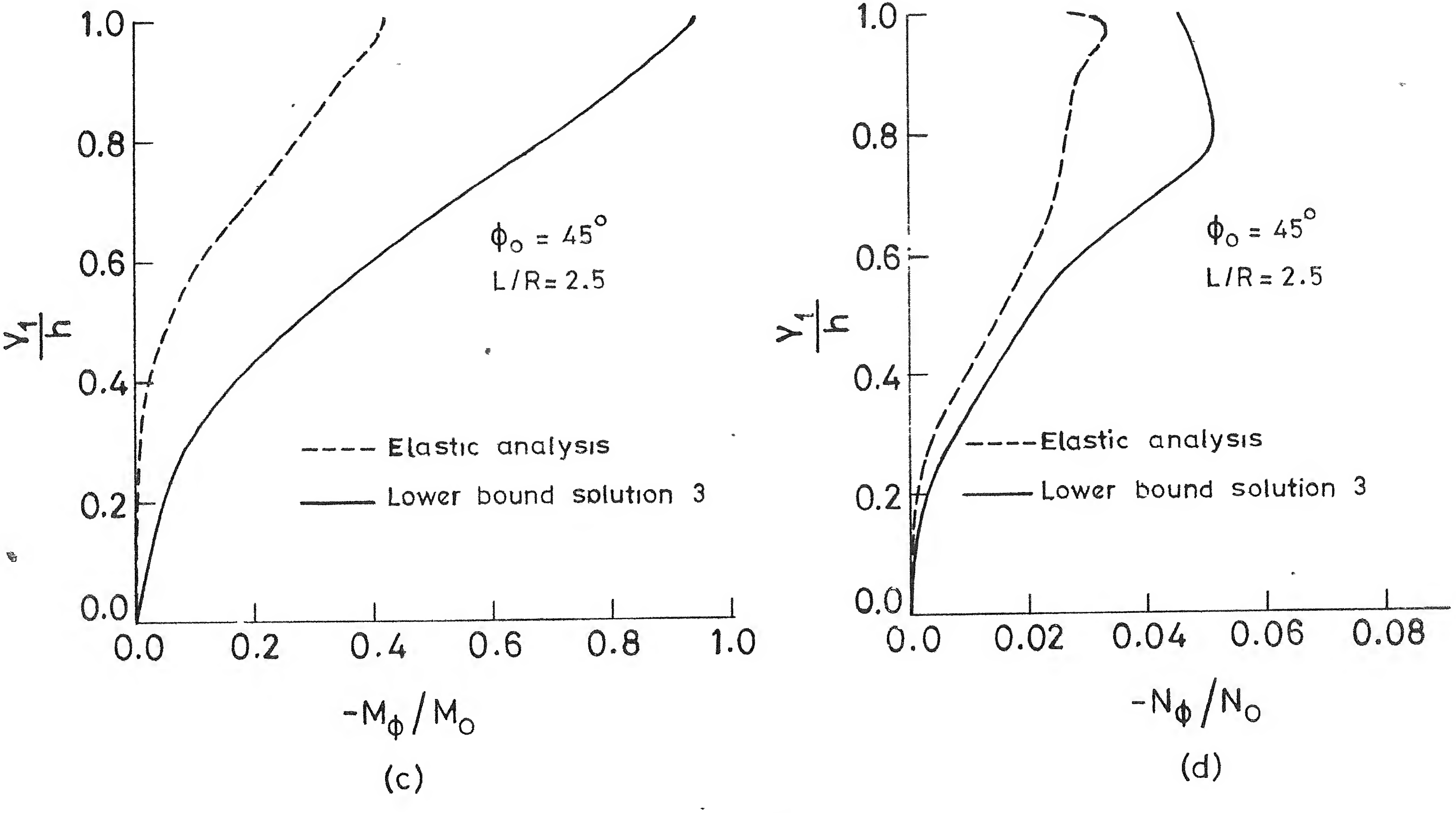


FIG. 3.4 DISTRIBUTION OF STRESS RESULTANTS-EXAMPLE 2

the stress resultants at the critical sections in reduced form are plotted for both elastic and limit analyses for purpose of comparison (Fig. 3.4). The stress resultants are tabulated in Table 3.3.

For a comparative study the values of the lower bound computed from the three solutions, are presented in the Table 3.4, for the two shells. In each solution the collapse load p is computed using both N_c and M_ϕ-N_ϕ yield criteria and the smaller of these two values is taken as the lower bound. In like manner the lower bounds are determined by the three solutions and the best lower bound is indicated.

3.9 Design of Shells

In this section the design procedure for a simply supported RC cylindrical shell roof with free longitudinal edges is described. Two shells are designed one for each mode of collapse. Elastic design is also carried out for working loads.

The step by step procedure of design is as follows :

(a) For a given working load, the design ultimate load will be estimated using suitable load factors. For simplicity of calculations, distribution of live load can be taken as gravity loading.

(b) The value of β will be preassigned. β_1 value will be taken from Table 3.1.

(c) The nondimensional load p_o will be estimated from the three lower bound solutions or from graphs (Fig. 3.2). Out of these three values the maximum value will be picked up. The lower bound solution that yields the maximum value of p_o will be noted.

(d) Knowing p_o, the ultimate load will be calculated from the relation $p_o = \frac{p}{\sigma_c}$ where p is the ultimate load. Depending on the relative values of the calculated and design ultimate loads σ_c and t will be revised.

(e) The stress resultants will be computed, using the corresponding lower bound solution.

(f) The transverse steel will be found from the Eq. (2.29) so as to follow the force distribution at different points on the shell.

(g) Longitudinal and diagonal reinforcement will be computed from the distributions of N_x and $N_{x\phi}$ respectively.

In all these calculations, the stress in the steel will be the yield stress σ_{sy}.

To compare with the elastic design, two shells with different geometric parameters are designed. Elastic design is carried out for a total working load of 290 kg/m^2 and limit design is done with an average load factor of 2.0.

Shell 1 fails due to N_c-yield criterion and solution 2 gives the best lower bound. Shell 2, fails due to M_ϕ-N_ϕ yield criterion and solution 3 gives the best lower bound. The parameters of the two shells are ;

Shell 1 (Long Shell) : $\phi_o = 30^o$, $\beta = \frac{2}{3}\phi_o$, $\frac{L}{R} = 3.333$ and

$R = 8.0$ m.

Shell 2 (Medium Shell) : $\phi_o = 45^o$, $\beta = \frac{2}{3}\phi_o$, $\beta_1 = 0.3818$,

$\frac{L}{R} = 2.5$ and $R = 8.0$ m.

For the two shells, thickness of the shell, compressive strength of concrete and the quantity of steel at critical sections are presented in the Table 3.5. The quantity of longitudinal steel shown indicates the steel required to resist the total longitudinal tension developed at the centre of the span. The transverse steel is estimated at crown to resist M_ϕ and N_ϕ per unit length. Diagonal steel per unit length is computed for resisting the diagonal tension equal to $N_{x\phi}$ developed at support.

From the Table 3.5 it can be seen that for shell 1, which fails due to N_c-yield criterion there is a considerable economy due to the reduction in the thickness and strength of concrete in the limit design as compared to the elastic design. There is a saving in steel of 15 per cent in the

Table 3.5 – Comparison between elastic and limit designs

	Thickness t in cms	f_c in kg/cm^2	Transverse steel in cm^2/m	Longitudinal steel in cm^2	Diagonal steel in cm^2/m
Shell 1 (Long Shell)					
a) Elastic design	10.0	200.0	8.8	102.0	11.5
b) Limit design	8.0	150.0	6.0	86.0	17.0
Shell 2 (Medium Shell)					
a) Elastic design	10.0	200.0	9.8	51.0	8.6
b) Limit design	9.0	150.0	11.0	65.0	11.5

longitudinal and 33 per cent in the transverse directions but the increase in the diagonal reinforcement is 30 per cent.

For the shell 2, which fails due to M_ϕ-N_ϕ yield criterion, there is an economy in the concrete due to reduction in the thickness of shell and the strength of concrete over elastic design. But in the limit design there is a considerable increase of the amount of steel. The increase is 27.5 per cent in the longitudinal direction, 12 per cent in the transverse direction and 35 per cent in the diagonal steel. Hence the total economy depends on the relative costs of the materials.

These figures gives only a comparative idea but not absolute values as these are computed at the critical sections only.

From the foregoing analysis of shells it is concluded that long shells fail due to the N_c-yield criterion and short shells fail due to M_ϕ-N_ϕ yield criterion.

CHAPTER 4

LIMIT ANALYSIS OF CYLINDRICAL SHELL ROOFS WITH LONGITUDINAL EDGE BEAMS

4.1 General

The load carrying capacity of a shell can be increased either by providing edge beams or by increasing the thickness of the shell. In the case of short shells the latter proposition is economical while the former proposition for medium and long shells. This is more so in the case of long shells. With this in view lower bound solutions are derived for simply supported RC cylindrical shell roofs with longitudinal edge beams subjected to uniform gravity loading.

Three lower bound solutions are developed using the same procedure presented in the previous chapter. Two shells one long and other medium, with different geometric parameters are considered. The elastic analysis is also carried out for working loads. The distribution of stress resultants at critical sections are presented graphically for a comparative study. Two shells are also designed by both elastic and proposed methods and relative economy is discussed.

4.2 Lower Bound Solutions

The lower bound solutions for the class of shells under consideration have to satisfy the equilibrium equations,

boundary conditions and the yield conditions. The equilibrium equations and yield conditions are presented in Chapter 2 in Sections 2.2 and 2.3.6 respectively. The stress resultants developed in the shell have to satisfy boundary conditions at the junction of the shell with the edge beams. In other words the interacting forces developed at the interface of the shell and the edge beam should be compatible.

Consider a RC cylindrical shell with edge beam. The coordinate system is shown in Fig. 4.1a. The geometry of the shell and loading are symmetric about the X axis i.e., at $\phi = 0$ and the longitudinal edges are stiffened with edge beams at $\phi = \phi_0$. Here ϕ is the angle measured from the crown to any point on the shell across the section and ϕ_0 is the semicentral angle. The cross section of the shell is divided into two zones, viz. the top compression zone for $0 \leq \phi \leq \beta$ and bottom tension zone for $\beta \leq \phi \leq \phi_0$, where β is an angle defining the location of the neutral axis. The stress resultants have different expressions in the two zones but have to satisfy the continuity conditions at the region boundary of the two zones.

4.3 Equilibrium of the Edge Beam

The edge beam is in equilibrium under the action of its self weight, the longitudinal force developed in it and the interacting forces at the interface of the shell and the

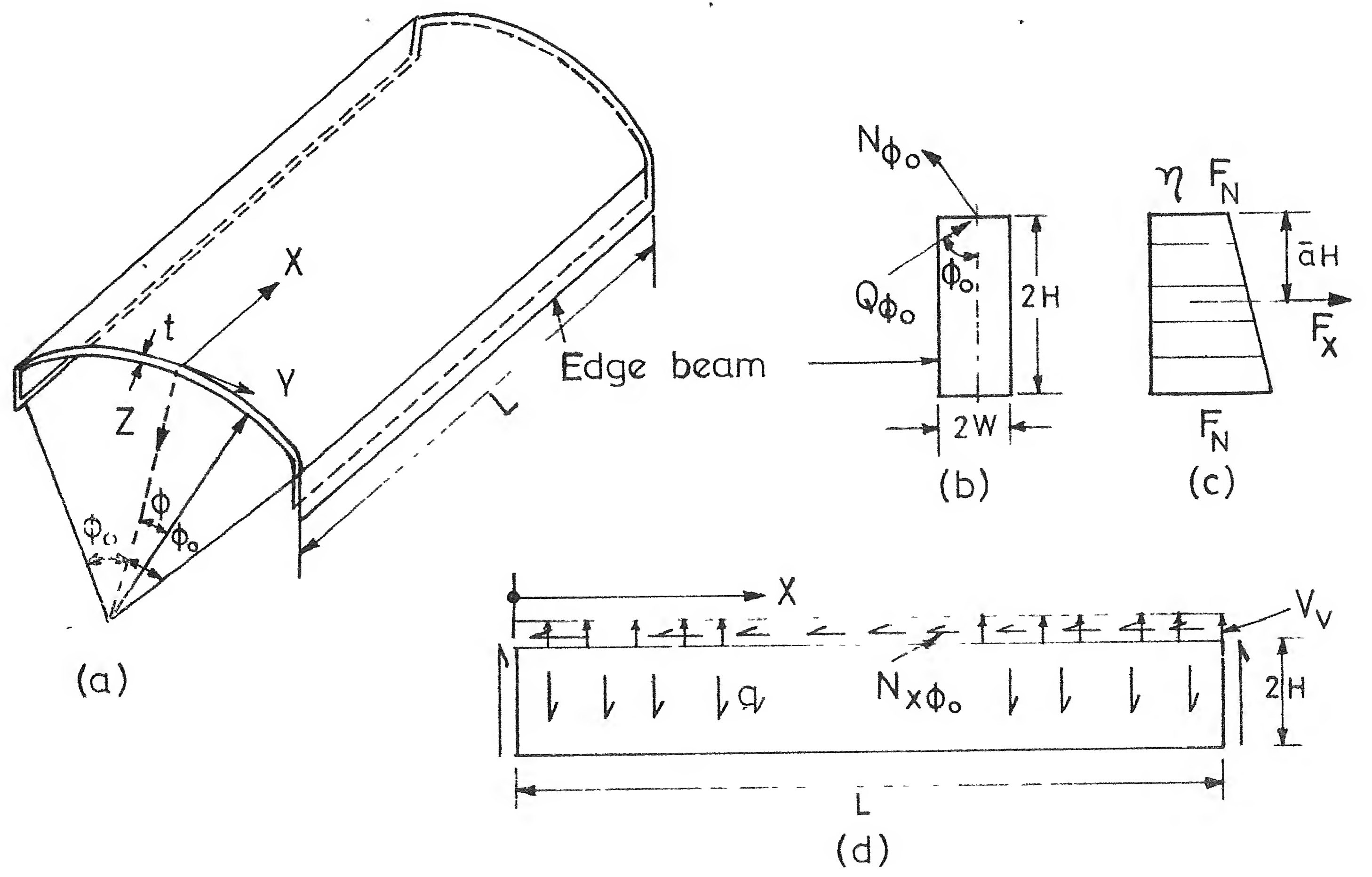


FIG 4.1 (a) COORDINATE SYSTEM (b) EDGE BEAM SHELL INTERACTING FORCES (c) EDGE BEAM FORCE DISTRIBUTION (d) EDGE BEAM LOADING

2H = depth of the edge beam,

2W = width of the edge beam.

From Fig. 4.1c it can be shown that

$$\bar{a} = \frac{2(2 + n)}{3(1 + n)}$$

Taking the variation of F_x in the longitudinal direction as parabolic,

$$F_x = \frac{4F_o}{L^2} (x) (L - x) . \qquad \ldots (4.1)$$

At any distance x from the origin, the equilibrium of the edge beam in the longitudinal direction gives that

$$\int_0^x N_{x\phi_o} \, dx = F_x . \qquad \ldots (4.2)$$

Differentiating both sides with respect to x

$$N_{x\phi_o} = \frac{d F_x}{dx} ,$$

$$= \frac{4 F_o}{L^2} (L - 2x) . \qquad \ldots (4.3)$$

Let V_v and V_h be the resultant vertical and horizontal components of the interacting forces acting on the edge beam. They are given by

$$V_v = N_{\phi_o} \sin \phi_o + Q_{\phi_o} \cos \phi_o, \qquad \ldots (4.4)$$

$$V_h = N_{\phi_o} \cos \phi_o - Q_{\phi_o} \sin \phi_o. \qquad \ldots (4.5)$$

Using the assumption (ii) Eq. (4.5) yields

$$Q_{\phi_o} = N_{\phi_o} \cot \phi_o . \qquad \ldots (4.6)$$

Eqs. (4.4) and (4.6) give

$$N_{\phi_o} = V_v \sin \phi_o . \qquad \ldots (4.7)$$

From Eqs. (4.6) and (4.7)

$$Q_{\phi_o} = V_v \cos \phi_o . \qquad \ldots (4.8)$$

Now the edge beam is in equilibrium under the action of q, $N_{x\phi_o}$ and V_v (Fig. 4.1d). Taking moments about the top of the edge beam, external bending moment M at $x = \frac{L}{2}$ is

$$M = \frac{qL^2}{8} - \frac{V_v L^2}{8} , \qquad \ldots (4.9)$$

and the internal resisting moment M_r at $x = \frac{L}{2}$ is

$$M_r = F_o \bar{a} H . \qquad \ldots (4.10)$$

Equating Eqs. (4.9) and (4.10) one gets

$$V_v = (q - \frac{8 F_o \bar{a} H}{L^2}) . \qquad \ldots (4.11)$$

Substitution of Eq. (4.11) in Eqs. (4.7) and (4.8) give the forces N_{ϕ_o} and Q_{ϕ_o} respectively.

4.4 Symmetry, Boundary and Continuity Conditions

The shell is symmetric in geometry and loading about X-axis i.e., at $\phi = 0$. Hence only half part of the shell is considered using the symmetry conditions at $\phi = 0$. As the cross section of the shell is divided into the top compression zone and bottom tension zone, the stress resultants have different expressions in the two regions. But the stress resultants have to satisfy continuity conditions at the region boundary. Lastly, the stress resultants have to satisfy the boundary conditions at $\phi = \phi_0$ i.e., at the junction of the shell with the edge beam. They are summarised as follows :

(i) Symmetry conditions :

$$N_{x\phi} = \frac{\partial N_\phi}{\partial \phi} = \frac{\partial M_\phi}{\partial \phi} = 0 \quad \text{at} \quad \phi = 0 \;. \qquad \ldots (4.12)$$

(ii) Continuity conditions :

$N_{x\phi}$, N_ϕ and M_ϕ and the derivatives $\frac{\partial N_\phi}{\partial \phi}$ and $\frac{\partial M_\phi}{\partial \phi}$ are continuous at $\phi = \beta$ . ... (4.13)

(iii) Boundary conditions :

The stress resultants $N_{x\phi}$, N_ϕ, Q_ϕ and M_ϕ at $\phi = \phi_0$ should be equal to $N_{x\phi_0}$, N_{ϕ_0}, Q_{ϕ_0} and zero respectively. Therefore using Eqs. (4.3), (4.7) and (4.8) it can be shown that at $\phi = \phi_0$

$$N_{x\phi} = \frac{4\ F_o}{L^2}(L - 2x)\ ,$$

$$N_{\phi} = V_v \sin \phi_o\ , \qquad \ldots (4.14)$$

$$Q_{\phi} = V_v \cos \phi_o\ ,$$

where V_v is given by Eq. (4.11).

The shell is considered as simply supported at the ends. At the simply supported ends $N_x = 0$ at $x = 0$ and $x = L$.

4.5 Lower Bound Solution 1

In this solution the distribution for N_x is assumed as uniform across the cross section of the shell both in the tension and compression zones as shown in Fig. 4.2b. As N_x is uniform across the section, it is independent of ϕ. This kind of distribution of N_x can be expected for long shell where the behaviour is predominantly of the beam type. When the N_c-yield criterion governs the failure (collapse), then the collapse mode of a long shell is similar to that of a beam.

Since the magnitude and longitudinal variation of N_x is unknown, it is evaluated from the static equilibrium conditions of the shell, considered as a simply supported beam subjected to uniform gravity loading of intensity p. Then the other stress resultants $N_{x\phi}$, N_{ϕ} , Q_{ϕ} and M_{ϕ} are obtained using the equilibrium equations (2.15) through (2.18).

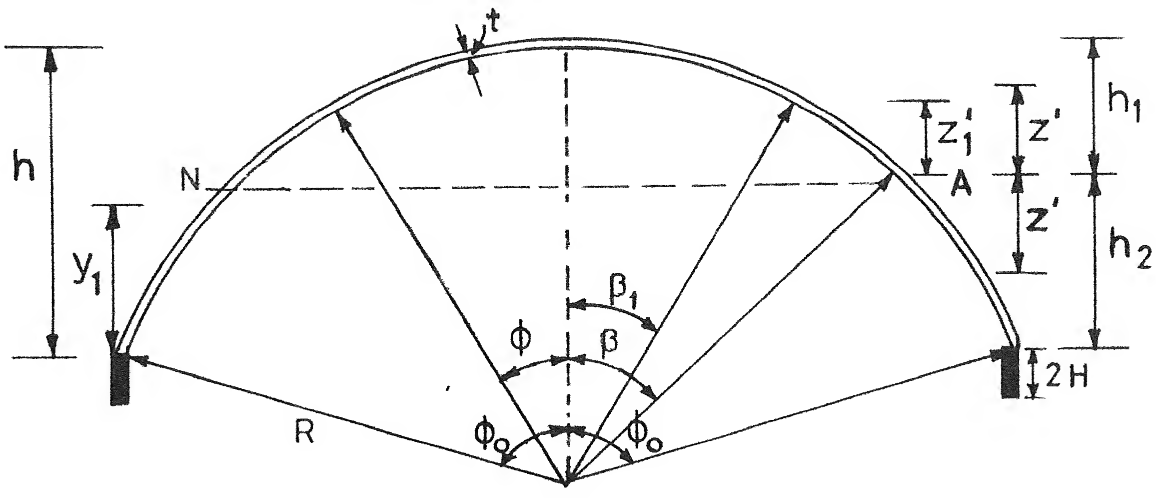


FIG.4.2 (a) SHELL SECTION

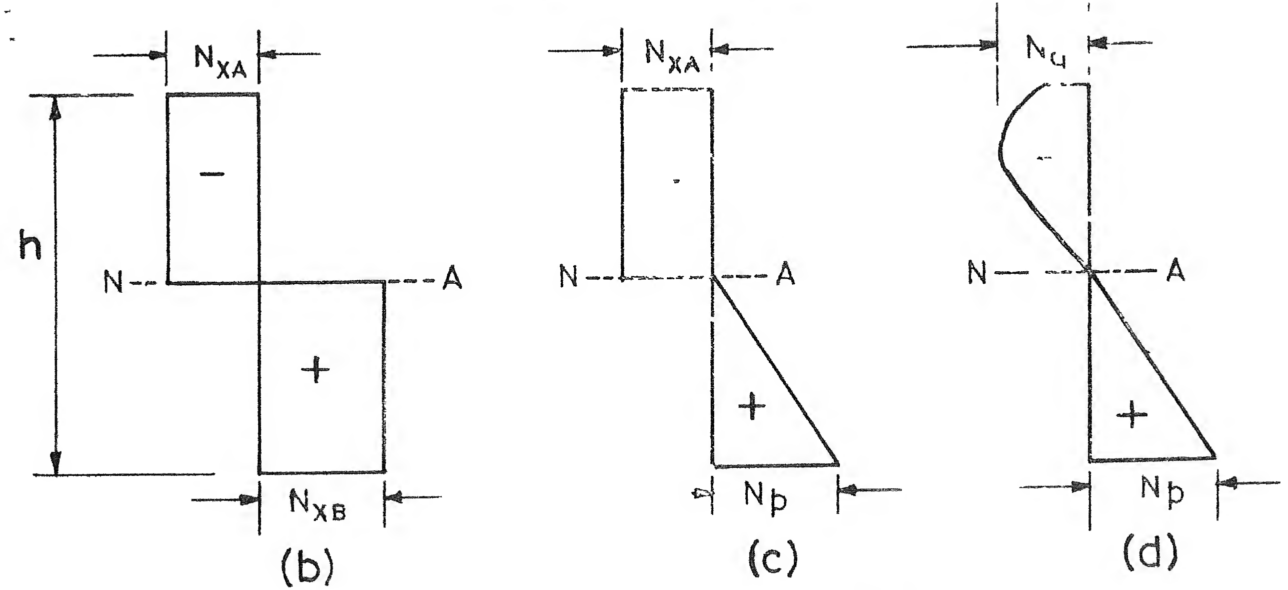


FIG.4.2 N_x DISTRIBUTION (b) SOLUTION 1 (c) SOLUTION 2 (d) SOLUTION 3

The longitudinal force developed in the edge beam is taken as uniform across the depth. In other words steel is distributed uniformly across the depth of the edge beam to produce uniform force per unit depth. Hence η is taken as equal to 1 (Fig. 4.1c).

4.5.1 Stress resultants

(i) Evaluation of N_x :

Let N_{XA} and N_{XB} represent the magnitudes of N_x in the compression and tension zones of the shell as shown in Fig. 4.2b. Thus

$$N_x = - N_{XA} \quad \text{for } 0 \leq \phi \leq \beta , \qquad \dots (4.15)$$

$$N_x = N_{XB} \quad \text{for } \beta \leq \phi \leq \phi_o . \qquad \dots (4.16)$$

Since N_x is uniform across the section, the resultant compressive and tensile forces act through the centroids of the corresponding parts of the shell on either side of the neutral axis. Let $\bar{h}_1$ and $\bar{h}_2$ be the vertical distances of the centroids of compressive and tensile regions from neutral axis respectively. They are given by

$$\bar{h}_1 = R \left[\frac{\sin \beta}{\beta} - \cos \beta \right] ,$$

$$\bar{h}_2 = R \left[\cos \beta - \frac{(\sin \phi_o - \sin \beta)}{(\phi_o - \beta)} \right] .$$

Addition of $\bar{h}_1$ and $\bar{h}_2$ gives

$$\bar{h}_1 + \bar{h}_2 = a_1 R,$$

where

$$a_1 = \frac{(\phi_o \sin\beta - \beta \sin\phi_o)}{(\phi_o - \beta)\beta} .$$

While considering the static equilibrium of the shell tensile and compressive forces are taken as positive and negative respectively. The static equilibrium condition (algebraic sum of the forces in the longitudinal direction is equal to zero) results in

$$2N_{XB} R(\phi_o - \beta) + 2F_x - 2N_{XA} R\beta = 0 . \qquad \dots (4.17)$$

where F_x is the longitudinal force in the edge beam.

The moment of resistance M_r at any section x is obtained by taking moments of the internal forces about the centroid of the compression zone. Thus

$$M_r = 2N_{XB} (\phi_o - \beta) a_1 R^2 + 2F_x a_2 R,$$

where

$$a_2 = \frac{\sin\beta}{\beta} - \cos\phi_o + \frac{H}{R} .$$

The external bending moment M at any section x subjected to uniformly distributed gravity loading of intensity p, is

$$M = pR\phi_o(x) (L - x) + q(x) (L - x) .$$

The second term on the right hand side is due to self weight of the edge beams.

Equating the moment of resistance M_r and the external bending moment M and substituting F_x from Eq. (4.1), the following result is obtained

$$N_{XB} = \frac{1}{2(\phi_o - \beta)a_1 R}\left(p\ \phi_o + \frac{q}{R} - 8F_o\ a_2\right)(x)(L - x). \qquad \ldots (4.18)$$

Substitution of Eqs. (4.1) and (4.18) in Eq. (4.17) yield

$$N_{XA} = \frac{1}{2\beta a_1 R}\left(p \quad {}_o + \frac{q}{R} - \frac{8a_3\ F_o}{L^2}\right)(x)\ (L - x), \qquad \ldots (4.19)$$

where

$$a_3 = \frac{1}{(\phi_o - \beta)}\left[\sin\ \phi_o - \sin\beta - (\phi_o - \beta)(\cos\ \phi_o - \frac{H}{R})\right].$$

From substitution of Eqs. (4.19) and (4.18) in Eqs. (4.15) and (4.16) respectively it is obtained

$$N_x = -\frac{1}{2\beta a_1 R}\left(p\ \phi_o + \frac{q}{R} - 8a_3\ \frac{F_o}{L^2}\right)(x)\ (L - x)$$

$$\text{for } 0 \leq \phi \leq \beta\ , \qquad \ldots (4.20)$$

$$N_x = \frac{1}{2(\phi_o - \beta)a_1 R}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o\ a_2}{L^2}\right)(x)\ (L - x)$$

$$\text{for } \beta \leq \phi \leq \phi_o\ . \qquad \ldots (4.21)$$

<u>Evaluation of $N_{x\phi}$</u>

The value of $N_{x\phi}$ is obtained as follows :

Eq. (2.15) can be written as

$$\frac{\partial N_{x\phi}}{\partial \phi} = - R \frac{\partial N_x}{\partial x} . \qquad \ldots (4.22)$$

(i) For $0 \leq \phi \leq \beta$

Substitution of Eq. (4.20) in Eq. (4.22) yields

$$\frac{\partial N_{x\phi}}{\partial \phi} = \frac{1}{2\beta a_1} (p \ \phi_o + \frac{q}{R} - \frac{8a_3 \ F_o}{L^2}) \ (L - 2x) \ . \qquad (4.23)$$

Integrating with respect to ϕ one gets

$$N_{x\phi} = \frac{\phi}{2\beta a_1} (p \ \phi_o + \frac{q}{R} - \frac{8a_3 \ F_o}{L^2}) \ (L - 2x) + D(x), \qquad \ldots (4.24)$$

where $D(x)$ is an arbitrary function of x only.

(ii) For $\beta \leq \phi \leq \phi_o$

Substituting Eq. (4.21) in Eq. (4.22) it is found

$$\frac{\partial N_{x\phi}}{\partial \phi} = - \frac{1}{2(\phi_o - \beta)a_1} (p \ \phi_o + \frac{q}{R} - \frac{8F_o \ a_2}{L^2}) \ (L - 2x), \quad \ldots (4.25)$$

which gives on integration with respect to ϕ

$$N_{x\phi} = - \frac{\phi}{2(\phi_o - \beta)a_1} (p \ \phi_o + \frac{q}{R} - \frac{8F_o \ a_2}{L^2}) \ (L - 2x) + E(x), \qquad \ldots (4.26)$$

where $E(x)$ is an arbitrary function of x only.

Evaluation of N_ϕ :

N_ϕ is determined in the following manner :

Eq. (3.16) is rearranged as

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = R^2 \frac{\partial^2 N_x}{\partial x^2} - 2pR \cos \phi \; . \qquad \ldots (4.27)$$

(i) For $0 \leq \phi \leq \beta$

Substituting for N_x (Eq. (4.20))

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = \frac{R}{\beta \; a_1} (p \; \phi_o + \frac{q}{R} - \frac{8a_3 \; F_o}{L^2}) - 2pR \cos \phi \; . \qquad \ldots (4.28)$$

The foregoing differential equation has the solution in the form

$$N_\phi = A_1 \sin \phi + B_1 \cos \phi - pR \; \phi \sin \phi + \frac{R}{\beta \; a_1}(p \; \phi_o + \frac{q}{R} - \frac{8a_3 \; F_o}{L^2}),$$

$$\ldots (4.29)$$

where A_1 and B_1 are constants of integration. On differentiation Eq. (4.29) gives

$$\frac{\partial N_\phi}{\partial \phi} = A_1 \cos \phi - B_1 \sin \phi - pR(\sin \phi + \phi \cos \phi) \; . \qquad \ldots (4.30)$$

(ii) For $\beta \leq \phi \leq \phi_o$

Eqs. (4.21) and (4.27) give

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = - \frac{R}{(\phi_o - \beta) a_1}(p \; \phi_o + \frac{q}{R} - \frac{8a_2 \; F_o}{L^2}) - 2pR \cos \phi \; .$$

$$\ldots (4.31)$$

The solution to the differential equation is

$$N_\phi = A_2 \sin\phi + B_2 \cos\phi - pR\,\phi \sin\phi - \frac{R}{(\phi_o - \beta)a_1} (p\,\phi_o +$$

$$\frac{q}{R} - \frac{8a_2 F_o}{L^2}) \;, \qquad \ldots (4.32)$$

where A_2 and B_2 are constants of integration. Differentiating one gets

$$\frac{\partial N_\phi}{\partial\phi} = A_2 \cos\phi - B_2 \sin\phi - pR(\sin\phi + \phi\cos\phi). \qquad \ldots (4.33)$$

Evaluation of M_ϕ :

M_ϕ is evaluated as follows :

Eq. (2.17) is rewritten as

$$\frac{\partial^2 M_\phi}{\partial\phi^2} = -N_\phi R - pR^2 \cos\phi \;. \qquad \ldots (4.34)$$

(i) For $0 \leq \phi \leq \beta$

Substitution of Eq. (4.29) in Eq. (4.34) yields

$$\frac{\partial^2 M_\phi}{\partial\phi^2} = -A_1 R \sin\phi - B_1 R \cos\phi + pR^2(\phi\sin\phi - \cos\phi) -$$

$$\frac{R^2}{\beta a_1}(p\,\phi_o + \frac{q}{R} - \frac{8a_3 F_o}{L^2}) \;. \qquad \ldots (4.35)$$

On integrating once it gives

$$\frac{\partial M_\phi}{\partial \phi} = A_1 R \cos \phi - B_1 R \sin \phi - pR^2 \phi \cos \phi - \frac{R^2 \phi}{\beta a_1}(p \phi_o + \frac{q}{R} - \frac{8a_3 F_o}{L^2}) + C_1 , \qquad \ldots (4.36)$$

where C_1 is a constant of integration. Integrating once again

$$M\phi = A_1 R \sin \phi + B_1 R \cos \phi - pR^2 (\cos \phi + \phi \sin \phi) - \frac{R^2 \phi^2}{2\beta a_1} (p \phi_o + \frac{q}{R} - \frac{8a_3 F_o}{L^2}) + C_1 \phi + C_2 , \qquad \ldots (4.37)$$

where C_2 is a constant of integration.

(ii) <u>For $\beta \leq \phi \leq \phi_o$</u>

From Eqs. (4.32) and (4.34) it is found

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = - A_2 R \sin \phi - B_2 R \cos \phi + pR^2(\phi \sin \phi - \cos \phi) + \frac{R^2}{(\phi_o - \beta) a_1} (p \phi_o + \frac{q}{R} - \frac{8a_2 F_o}{L^2}) . \qquad \ldots (4.38)$$

Integrating once

$$\frac{\partial M_\phi}{\partial \phi} = A_2 R \cos \phi - B_2 R \sin \phi - pR^2 \phi \cos \phi + \frac{R^2 \phi}{(\phi_o - \beta) a_1}(p \phi_o + \frac{q}{R} - \frac{8a_2 F_o}{L^2}) + C_3 , \qquad \ldots (4.39)$$

where C_3 is a constant of integration. Integrating once again

$$M_\phi = A_2 R \sin\phi + B_2 R \cos\phi - pR^2 (\cos\phi + \phi \sin\phi)$$
$$+ \frac{R^2 \phi^2}{2(\phi_o - \beta) a_1} (p \phi_o + \frac{q}{R} - \frac{8a_2 F_o}{L^2}) + C_3 \phi + C_4, \quad \ldots (4.40)$$

where C_4 is a constant of integration.

<u>Evaluation of Q_ϕ</u> :

Q_ϕ is obtained as explained below :

Eq. (2.18) may be written as

$$Q_\phi = \frac{1}{R} \frac{\partial M_\phi}{\partial \phi} . \quad \ldots (4.41)$$

Substitution of Eqs. (4.36) and (4.39) in Eq. (4.41) yield

$$Q_\phi = A_1 \cos\phi - B_1 \sin\phi - pR\phi \cos\phi - \frac{R\phi}{\beta a_1} (p \phi_o + \frac{q}{R} - \frac{8a_3 F_o}{L^2}) + \frac{C_1}{R}, \quad \text{for } 0 \le \phi \le \beta \quad \ldots$$

$$Q_\phi = A_2 \cos\phi - B_2 \sin\phi - pR\phi \cos\phi + \frac{R\phi}{(\phi_o - \beta) a_1} (p \phi_o + \frac{q}{R} - \frac{8a_2 F_o}{L^2}) + \frac{C_3}{R}, \quad \text{for } \beta \le \phi \le \phi_0 \quad \ldots$$

respectively.

The constants of integration are written in the modified form as follows :

$$A_1 = A_{11} pR + A_{12} q + 8A_{13} \frac{F_o R}{L^2},$$

$$B_1 = B_{11} pR + B_{12} q + 8B_{13} \frac{F_o R}{L^2},$$

$$A_2 = A_{21} pR + A_{22} q + 8A_{23} \frac{F_o R}{L^2},$$

$$B_2 = B_{21}\, pR + B_{22}\, q + 8\, B_{23} \frac{F_o R}{L^2} ,$$

$$C_1 = C_{11}\, pR^2 + C_{12}\, qR + 8\, C_{13} \frac{F_o R^2}{L^2} ,$$

$$C_2 = C_{21}\, pR^2 + c_{22}\, qR + 8\, C_{23} \frac{F_o R^2}{L^2} ,$$

$$C_3 = C_{31}\, pR^2 + C_{32}\, qR + 8\, C_{23} \frac{F_o R^2}{L^2} ,$$

$$C_4 = C_{41}\, pR^2 + C_{42}\, qR + 8\, C_{43} \frac{F_o R^2}{L^2} .$$

The constants of integration are determined from the conditions (4.12) through (4.14). Thus

$$D(x) = 0$$

$$E(x) = \frac{\phi_o}{2a_1(\phi_o - \beta)} \left(p\, \phi_o + \frac{q}{R} - \frac{8F_o a_2}{L^2}\right)(L-2x) + \frac{4F_o}{L^2}(L - 2x)$$

$$A_{11} = A_{12} = A_{13} = 0$$

$$A_{21} = \frac{\phi_o^2 \sin\beta}{a_1\beta(\phi_o - \beta)}$$

$$A_{22} = \frac{A_{21}}{\phi_o}$$

$$A_{23} = \frac{\sin\beta}{a_1\beta(\phi_o - \beta)} \left(\phi_o \cos\phi_o - \sin\phi_o - \phi_o \frac{H}{R}\right)$$

$$B_{11} = \frac{\phi_o(\beta \cos\phi_o - \phi_o \cos\beta)}{a_1\beta(\phi_o - \beta)}$$

$$B_{12} = \frac{B_{11}}{\phi_o}$$

$$B_{13} = -\frac{1}{a_1\beta(\phi_o-\beta)}\Big[\frac{H}{R}(\beta\ \cos\ \phi_o - \phi_o\ \cos\beta) + \phi_o\ \cos\ (\phi_o - \beta)$$

$$- \sin\ (\phi_o-\beta) - \beta\Big]$$

$$B_{21} = \frac{\phi_o\ \cos\ \phi_o}{a_1(\phi_o - \beta)}$$

$$B_{22} = \frac{B_{21}}{\phi_o}$$

$$B_{23} = -\frac{(\frac{H}{R}\beta\ \cos\ \phi_o + \sin\beta\ \ \cos\ \phi_o + \phi_o\ \sin\beta\ \sin\ \phi_o - \beta)}{\beta\ a_1(\phi_o - \beta)}$$

$$C_{11} = C_{12} = C_{13} = 0$$

$$C_{21} = \Big[\cos\ \phi_o + \frac{\phi_o\ (2 + \phi_o\beta)}{2\beta a_1}\Big]$$

$$C_{22} = \frac{(\phi_o\beta + 2)}{2\beta a_1} - \sin\ \phi_o$$

$$C_{23} = \frac{1}{a_1(\phi_o - \beta)\beta}\Big[\frac{H}{R}(\phi_o\ \sin\beta\ \sin\ \phi_o + \frac{\beta}{2}\ \cos\ 2\phi_o + \frac{\beta}{2})$$

$$+ (\phi_o-\beta)\ \cos\ \phi_o - \sin\ \phi_o + \sin\beta\Big] - \phi_o(1 - \frac{a_2}{2a_1}) - \frac{\beta}{2}$$

$$C_{31} = \frac{\phi_o^2}{a_1(\phi_o - \beta)}$$

$$C_{32} = \frac{\phi_o}{a_1(\phi_o - \beta)}$$

$$C_{33} = \frac{\phi_o\ a_2}{a_1(\phi_o - \beta)} - 1$$

$$C_{41} = \frac{1}{2a_1\beta(\phi_o-\beta)}\left[\phi_o\beta(\phi_o^2 - 2) + 2\phi_o \sin\beta \cos\phi_o - \beta \sin 2\phi_o\right]$$

$$C_{42} = \frac{1}{2a_1\beta(\phi_o-\beta)}\left[\phi_o^2\beta - 2\phi_o \sin\beta \sin\phi_o - \beta(1 + \cos 2\phi_o)\right]$$

$$C_{43} = \frac{1}{a_1\beta(\phi_o-\beta)}\left[\sin\beta - \beta\cos\phi_o + \frac{H}{R}(\phi_o \sin\phi_o \sin\beta + \frac{\beta}{2}(1 + \cos 2\phi_o))\right] - \frac{a_2\phi_o^2}{2a_1(\phi_o-\beta)}$$

The stress resultants are summarised after substituting the constants of integration in the modified form as follows :

(i) <u>For $0 \leq \phi \leq \beta$</u>

$$N_x = \frac{1}{2\beta a_1 R}\left(p\phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2}\right)(x)(L - x) \qquad \ldots (4.42)$$

$$N_{x\phi} = \frac{\phi}{2\beta a_1}\left(p\phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2}\right)(L - 2x) \qquad \ldots (4.43)$$

$$N_\phi = -pR\left(\phi\sin\phi - \frac{\phi_o}{\beta a_1} - B_{11}\cos\phi\right) + q\left(B_{12}\cos\phi + \frac{1}{\beta a_1}\right) + \frac{8F_o R}{L^2}\left(B_{13}\cos\phi - \frac{a_3}{\beta a_1}\right) \qquad \ldots (4.44)$$

$$M_\phi = -pR^2\left(\frac{\phi^2\phi_o}{2\beta a_1} + \cos\phi + \phi\sin\phi - C_{21} - B_{11}\cos\phi\right) - qR\left(\frac{\phi^2}{2\beta a_1} - B_{12}\cos\phi - C_{22}\right) + \frac{8F_o R^2}{L^2}\left(\frac{\phi^2 a_3}{2\beta a_1} + B_{13}\cos\phi + C_{23}\right) \qquad \ldots (4.45)$$

$$Q_\phi = -pR\left(\frac{\phi_o\phi}{\beta a_1} + \phi\cos\phi + B_{11}\sin\phi\right) - q\left(\frac{\phi}{\beta a_1} + B_{12}\sin\phi\right)$$

$$+ \frac{8F_o R}{L^2}\left(\frac{a_3\phi}{\beta a_1} - B_{13}\sin\phi\right) \qquad \ldots (4.46)$$

(ii) For $\beta \leq \phi \leq \phi_o$

$$N_x = \frac{\left(p\,\phi_o + \frac{q}{R} - \frac{8F_o\,a_2}{L^2}\right)}{2(\phi_o - \beta)\,a_1 R}(x)(L - x) \qquad \ldots (4.47)$$

$$N_{x\phi} = \left[\frac{\left(p\,\phi_o + \frac{q}{R} - \frac{8F_o\,a_2}{L^2}\right)}{2a_1(\phi_o - \beta)}(\phi_o - \phi) + \frac{4F_o}{L^2}\right](L - 2x) \qquad \ldots (4.48)$$

$$N_\phi = -pR\left(\phi\sin\phi + \frac{\phi_o}{a_1(\phi_o - \beta)} - A_{21}\sin\phi - B_{21}\cos\phi\right)$$

$$+ q\left(A_{22}\sin\phi + B_{22}\cos\phi - \frac{1}{a_1(\phi_o - \beta)}\right)$$

$$+ \frac{8F_o R}{L^2}\left(\frac{a_2}{a_1(\phi_o - \beta)} + A_{23}\sin\phi + B_{23}\cos\phi\right) \qquad \ldots (4.49)$$

$$M_\phi = -pR^2\left(\cos\phi + \phi\sin\phi - \frac{\phi^2\,\phi_o}{2a_1(\phi_o - \beta)} - A_{21}\sin\phi\right.$$

$$\left. - B_{21}\cos\phi - C_{31}\phi - C_{41}\right) + qR\left(A_{22}\sin\phi\right.$$

$$\left. + B_{22}\cos\phi + \frac{\phi^2}{2a_1(\phi_o - \beta)} + C_{32}\phi + C_{42}\right)$$

$$+ \frac{8F_o R^2}{L^2}\left(A_{23}\sin\phi + B_{23}\cos\phi + C_{33}\phi + C_{43} - \frac{a_2\,\phi^2}{2a_1(\phi_o - \beta)}\right)$$

$$\ldots (4.50)$$

$$Q_\phi = -\,pR\left(\phi\cos\phi - \frac{\phi\,\phi_o}{a_1(\phi_o-\beta)} - A_{21}\cos\phi + B_{21}\sin\phi - C_{31}\right)$$

$$+\,q\left(A_{22}\cos\phi - B_{22}\sin\phi + \frac{\phi}{a_1(\phi_o-\beta)} + C_{32}\right)$$

$$+\,\frac{8F_o\,R}{L^2}\left(A_{23}\cos\phi - B_{23}\sin\phi - \frac{a_2\,\phi}{a_1(\phi_o-\beta)} + C_{33}\right) \qquad \ldots (4.51)$$

From the expressions for N_ϕ, M_ϕ and Q_ϕ it can be found that they **are** independent of x.

4.5.2 Determination of Limit Load

The limit load is evaluated by using the yield conditions given by Eqs. (2.26) and (2.30). Let p_{L1} and p_{L2} represent the two lower bounds obtained from the N_c and M_ϕ-N_ϕ yield criteria respectively. They are evaluated as follows :

(i) Due to N_c-yield criterion. The maximum value of N_x in compression is determined from Eq. (4.42) which occurs at $x = \frac{L}{2}$. Thus

$$N_{x\,max} = -\,\frac{\left[p\,\phi_o + \frac{q}{R} - \frac{8F_o\,a_3}{L^2}\right]L^2}{8\beta\,a_1\,R} \qquad \ldots (4.52)$$

From the yield condition (2.26), it is found

$$p_{L1} = \frac{8a_1\,\beta\,R\,N_c}{\phi_o\,L^2} - \frac{q}{\phi_o R} + \frac{8F_o\,a_3}{\phi_o\,L^2} \qquad \ldots (4.53)$$

(ii) Due to $M_\phi - N_\phi$ yield criterion : The failure of a cylindrical shell occurs in the transverse direction, if the yield condition given by Eq. (2.30) is satisfied. In a simply supported cylindrical shell with edge beams, N_ϕ is always compressive but M_ϕ may change in sign with in the range from 0 to ϕ_o. Therefore the yield condition is to be checked within the region 0 to ϕ_o. Since M_ϕ and N_ϕ are independent of x, it can be understood that if once yielding is initiated at a point, failure occurs for the entire length of the shell. In this limit analysis of shells it is observed that when the shell fails due to the $M_\phi - N_\phi$ yield criterion, the failure of the shell occurs at the crown. The absolute maximum value of M_ϕ also occurs at crown. Consequently the limit load due to this criterion can be found using the values of M_ϕ and N_ϕ at crown. Let $N_{\phi c}$ and $M_{\phi c}$ represent the values of N_ϕ and M_ϕ at crown, which are obtained from the Eqs. (4.44) and (4.45). Thus

$$N_{\phi c} = pR(\frac{\phi_o}{\beta a_1} + B_{11}) + q(B_{12} + \frac{1}{\beta a_1}) + \frac{8F_oR}{L^2}(B_{13} - \frac{a_3}{\beta a_1}), \quad \ldots (4.54)$$

$$M_{\phi c} = pR^2(B_{11} + C_{21} - 1) + qR(B_{12} + C_{22}) + \frac{8F_oR^2}{L^2}(B_{13} + C_{23}) \cdot \quad \ldots (4.55)$$

Substitution of $N_{\phi c}$ and $M_{\phi c}$ in Eq. (2.30) yields the collapse load

$$p_{L2} = \frac{1}{D_1}\left[b - \frac{qR}{M_o}(B_{12} + C_{22}) + \frac{cq}{N_o}\left(\frac{1}{\beta a_1} + B_{12}\right) - \frac{8F_o R^2}{M_o L^2}(B_{13} + C_{23}) + \frac{8c F_o R}{N_o L^2}\left(B_{13} - \frac{a_3}{\beta a_1}\right)\right] \quad \ldots (4.56)$$

wh re

$$D_1 = \frac{R^2}{M_o}(B_{11} + C_{21} - 1) - \frac{cR}{N_o}\left(B_{11} + \frac{\phi_o}{\beta a_1}\right).$$

The collapse load p is the smaller of the two values p_{L1} and p_{L2}.

4.6 Lower Bound Solution 2

In the lower bound solution 2, the distribution for N_x is as shown in Fig. 4.2c. In this, N_x is taken as uniform across the cross section in the compression zone and it varies linearly with depth z' from the neutral axis in the tension zone. The reasons for selecting this distribution are same as that for shells with free longitudinal edges (Section 3.5).

The variation of force developed in the edge beam is trapezoidal as indicated in the Fig. 4.1c. The magnitude of the force per unit depth at top of the edge beam is η times that of the one at the bottom.

4.6.1 Stress resultants :

The stress resultants N_x, $N_{x\phi}$, N_ϕ, M_ϕ and Q_ϕ are determined as follows :

Evaluation of N_x :

The magnitude and longitudinal variation of N_x is found from the static equilibrium conditions of the shell.

Let $\bar{h}_1$ denote the vertical distance of the centre of total compression from the neutral axis and it coincides with the centroid of the circular arc of the shell above neutral axis. Thus

$$\bar{h}_1 = R\left(\frac{\sin\beta}{\beta} - \beta\right) .$$

The depths of the compression and tension zones are represented by h_1 and h_2 respectively. They are given by

$$h_1 = R(1 - \cos\beta)$$

$$h_2 = R(\cos\beta - \cos\phi_0).$$

If z' is the vertical distance from the neutral axis to any point on the cross section of the shell, then z' is given (Fig. 4.2a) as

$$z' = R(\cos\phi - \cos\beta) \quad \text{for } 0 \leq \phi \leq \beta,$$

$$z' = R(\cos\beta - \cos\phi) \quad \text{for } \beta \leq \phi \leq \phi_0.$$

The maximum value of N_x in the tension zone and the uniform value in the compression zone are represented by N_p and N_{XA} respectively (Fig. 4.2c). Then the distribution of N_x in the two zones is expressed as

$$N_x = - N_{XA} \qquad \text{for } 0 \leq \phi \leq \beta ,$$

$$N_x = \frac{N_p z'}{h_2} = \frac{N_p(\cos\beta - \cos\phi)}{(\cos\beta - \cos\phi_o)} \qquad \text{for } \beta \leq \phi \leq \phi_o \qquad \ldots (4.57)$$

The force in the edge beam F_x at any section x is given by Eq. (4.1).

Taking tension and compression as positive and negative respectively, the static equilibrium condition of the shell, i.e., $\Sigma N_x \, ds = 0$ gives

$$2 \int_0^{\beta} N_x R \, d\phi + 2 \int_{\beta}^{\phi_o} N_x R \, d\phi + 2F_x = 0 \; .$$

Substitution of Eqs. (4.57) and integration yields

$$- N_{XA} R\beta + N_p R \frac{\left[(\phi_o - \beta) \cos\beta - \sin\phi_o + \sin\beta \right]}{(\cos\beta - \cos\phi_o)} + F_x = 0 \; ,$$

from which the following is obtained

$$N_p = \frac{N_{XA}\beta}{a_2} (\cos\beta - \cos\phi_o) - \frac{F_x}{Ra_2} (\cos\beta - \cos\phi_o) . \qquad \ldots (4.58)$$

where

$$a_2 = (\phi_o - \beta) \cos\beta - \sin\phi_o + \sin\beta \; .$$

Taking moments about the neutral axis, the moment of resistance M_r is obtained. Thus

$$M_r = 2\int_0^{\beta} N_x R\, d\phi\, z' + 2\int_{\beta}^{\phi_o} N_x R\, d\phi\, z' + 2F_x\,(h_2 + \bar{a}H).$$

Substitution of N_x from Eqs. (4.57) in the foregoing equation gives

$$M_r = 2N_{XA}\, a_1\, R^2 + 2F_x\, a_3\, R \qquad \ldots (4.59)$$

where

$$a_1 = (\sin\beta - \beta\cos\beta) + \frac{\beta}{a_2}\Big[\frac{(\phi_o - \beta)}{2}(2 + \cos 2\beta)$$

$$+ \frac{1}{4}(\sin 2\phi_o - \sin 2\beta) - 2\cos\beta(\sin\phi_o - \sin\beta)\Big],$$

$$a_3 = (\cos\beta - \cos\phi_o + \bar{a}\frac{H}{R}) - \frac{1}{a_2}\Big[\frac{(\phi_o - \beta)}{2}(2 + \cos 2\beta)$$

$$+ \frac{1}{4}(\sin 2\phi_o - \sin 2\beta) - 2\cos\beta(\sin\phi_o - \sin\beta)\Big].$$

The external bending moment M at any section x is

$$M = (pR\,\phi_o + q)\,(x)\,(L - x)\ . \qquad \ldots (4.60)$$

Equating Eqs. (4.59) and (4.60) and substitution of F_x from Eq. (4.1) one gets

$$N_{XA} = \frac{1}{2a_1 R}\,(p\,\phi_o + \frac{q}{R} - \frac{8F_o\, a_3}{L^2})\,(x)\,(L - x). \qquad \ldots (4.61)$$

The first part of Eq. (4.57) and Eq. (4.61) give

$$N_x = -\frac{1}{2a_1 R}(p\,\phi_o + \frac{q}{R} - \frac{8F_o\, a_3}{L^2})\,(x)(L - x) \text{ for } 0 \le \phi \le \beta.$$

$$\ldots (4.62)$$

Similarly using Eqs. (4.1), the second part of Eq. (4.57), (4.58) and (4.61) it is found that

$$N_x = \frac{\beta}{2a_1 a_2 R}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o a_4}{L^2}\right)(\cos\beta - \cos\phi)\ (x)\ (L - x)$$

$$\text{for } \beta \leq \phi \leq \phi_o \qquad \ldots (4.63)$$

where

$$a_4 = \frac{a_1}{\beta} + a_3.$$

Evaluation of $N_{x\phi}$:

$N_{x\phi}$ is determined as follows :

Eq. (2.15) yields

$$\frac{\partial N_{x\phi}}{\partial\phi} = -R\,\frac{\partial N_x}{\partial x}. \qquad \ldots (4.64)$$

(i) For $0 \leq \phi \leq \beta$

Substitution of Eq. (4.62) in Eq. (4.64) gives

$$\frac{\partial N_{x\phi}}{\partial\phi} = \frac{1}{2a_1}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2}\right)(L - 2x).$$

On integration with respect to ϕ

$$N_{x\phi} = \frac{\phi}{2a_1}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2}\right)(L - 2x) + D(x) \qquad \ldots (4.65)$$

where D(x) is a function of x only.

(ii) For $\beta \leq \phi \leq \phi_o$

From the Eqs. (4.63) and (4.64) it is found that

$$\frac{\partial N_{x\phi}}{\partial \phi} = - \frac{\beta}{2a_1 a_2} (p \ \phi_o + \frac{q}{R} - \frac{8F_o \ a_4}{L^2}) (\cos \beta - \cos \phi) (L - 2x).$$

On integration

$$N_{x\phi} = - \frac{\beta}{2a_1 a_2} (p \ \phi_o + \frac{q}{R} - \frac{8F_o a_4}{L^2})(\phi \cos \beta - \sin \phi) (L - 2x) + E(x) \qquad \ldots (4.66)$$

where E(x) is a function of x only.

Evaluation of N_ϕ :

Eq. (2.16) may be rewritten as

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = R^2 \frac{\partial^2 N_x}{\partial x^2} - 2pR \cos \phi \ . \qquad \ldots (4.67)$$

(i) For $0 \leq \phi \leq \beta$

Substitution of Eq. (4.62) in the foregoing equation yields

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = \frac{R}{a_1} (p \ \phi_o + \frac{q}{R} - \frac{8F_o \ a_3}{L^2}) - 2pR \cos \phi \ . \qquad \ldots (4.68)$$

This differential equation has a solution in the following form

$$N_\phi = A_1 \sin \phi + B_1 \cos \phi + \frac{R}{a_1} (p \ \phi_o + \frac{q}{R} - \frac{8F_o \ a_3}{L^2}) - pR \ \phi \sin \phi, \qquad \ldots (4.69)$$

where A_1 and B_1 are constants of integration. On differentiation one gets

$$\frac{\partial N_\phi}{\partial \phi} = A_1 \sin\phi - B_1 \sin\phi - pR\,(\phi \cos\phi + \sin\phi). \quad \ldots (4.70)$$

(ii) <u>For $\beta \leq \phi \leq \phi_o$</u>

Substituting Eq. (4.63) in Eq. (4.67)

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = -\frac{\beta R}{a_1 a_2}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o a_4}{L^2}\right)(\cos\beta - \cos\phi) - 2pR\cos\phi. \quad \ldots (4.71)$$

The solution of the differential equation is

$$N_\phi = A_2 \sin\phi + B_2 \cos\phi - \frac{\beta R}{a_1 a_2}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o a_4}{L^2}\right)\left(\cos\beta - \frac{\phi \sin\phi}{2}\right) - pR\,\phi \sin\phi\,, \quad \ldots (4.72)$$

where A_2 and B_2 are constants of integration. Differentiating with respect to ϕ the following is obtained :

$$\frac{\partial N_\phi}{\partial \phi} = A_2 \cos\phi - B_2 \sin\phi + \frac{\beta R}{2 a_1 a_2}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o a_4}{L^2}\right)(\phi \cos\phi + \sin\phi) - pR\,(\phi\cos\phi + \sin\phi). \quad \ldots (4.73)$$

<u>Evaluation of M_ϕ :</u>

M_ϕ is evaluated as in the following manner :

Eq. (2.17) gives

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = -N_\phi R - pR^2 \cos \phi . \qquad \ldots (4.74)$$

(i) For $0 \leq \phi \leq \beta$

Substitution of Eq. (4.69) in Eq. (4.74) yields

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = -A_1 R \sin \phi - B_1 R \cos \phi - \frac{R^2}{a_1} (p \phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2})$$

$$+ pR^2 \phi \sin \phi - pR^2 \cos \phi . \qquad \ldots (4.75)$$

Integrating twice

$$\frac{\partial M_\phi}{\partial \phi} = A_1 R \cos \phi - B_1 R \sin \phi - \frac{\phi R^2}{a_1} (p \phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2})$$

$$- pR^2 \phi \cos \phi + C_1 , \qquad \ldots (4.76)$$

and

$$M_\phi = A_1 R \sin \phi + B_1 R \cos \phi - \frac{\phi^2 R^2}{2a_1} (p \phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2})$$

$$- pR^2 (\cos \phi + \phi \sin \phi) + C_1 \phi + C_2 , \qquad \ldots (4.77)$$

where C_1 and C_2 are constants of integration.

(ii) For $\beta \leq \phi \leq \phi_o$

From Eqs. (4.72) and (4.74) it follows that

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = -A_2 R \sin \phi - B_2 R \cos \phi + \frac{\beta R^2}{2a_1 a_2} (p \phi_o + \frac{q}{R} - 8F_o a_4)$$

$$(2 \cos \beta - \phi \sin \phi) + pR^2 \phi \sin \phi - pR^2 \cos \phi .$$

On integration

$$\frac{\partial M_\phi}{\partial \phi} = A_2\ R \cos\phi - B_2\ R \sin\phi + \frac{\beta R^2}{2a_1 a_2}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o a_4}{L^2}\right)$$

$$(2\phi \cos\beta \ - \sin\phi + \phi\cos\phi) - pR^2\phi\ \cos\phi + C_3$$

$$\dots (4.79)$$

where C_3 is a constant of integration. Integrating once again it is found that

$$M_\phi = A_2\ R\sin\phi + B_2\ R\cos\phi + \frac{\beta R^2}{2a_1 a_2}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o a_4}{L^2}\right)$$

$$(\phi^2 \cos\beta + 2\cos\phi + \phi\sin\phi) - pR^2(\cos\phi + \phi\sin\phi)$$

$$+ C_{3\phi} + C_4 \qquad \dots (4.80)$$

where C_4 is a constant of integration.

Evaluation of Q_ϕ :

Q_ϕ is determined as follows :

Eq. (2.18) may be written as

$$Q_\phi = \frac{1}{R}\ \frac{\partial M_\phi}{\partial \phi}\ . \qquad \dots (4.81)$$

Substitution of Eqs. (4.76) and (4.79) respectively in Eq. (4.81) gives

$$Q_\phi = A_1 \cos\phi - B_1 \sin\phi - \frac{\phi R}{a_1}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2}\right) - pR\ \phi\cos\phi$$

$$+ \frac{C_1}{R} \qquad \text{for } 0 \leq \phi \leq \beta\ , \qquad \dots (4.82)$$

and

$$Q_\phi = A_2 \cos\phi - B_2 \sin\phi + \frac{\beta R}{2a_1 a_2}\left(p\,\phi_o + \frac{q}{R} - \frac{8F_o a_4}{L^2}\right)$$

$$(2\phi\cos\beta - \sin\phi + \phi\cos\phi) - pR\,\phi\cos\phi + \frac{C_3}{R}$$

$$\text{for } \beta \leq \phi \leq \phi_o. \qquad \ldots (4.83)$$

The constants of integration in the modified form are presented as

$$A_1 = A_{11}\, pR + A_{12}\, q + 8A_{13} \frac{F_o R}{L^2}$$

$$B_1 = B_{11}\, pR + B_{12}\, q + 8B_{13} \frac{F_o R}{L^2}$$

$$A_2 = A_{21}\, pR + A_{22}\, q + 8A_{23} \frac{F_o R}{L^2}$$

$$B_2 = B_{21}\, pR + B_{22}\, q + 8B_{23} \frac{F_o R}{L^2}$$

$$C_1 = C_{11}\, pR^2 + C_{12}\, qR + 8C_{13} \frac{F_o R^2}{L^2}$$

$$C_2 = C_{21}\, pR^2 + C_{22}\, qR + 8C_{23} \frac{F_o R^2}{L^2}$$

$$C_3 = C_{31}\, pR^2 + C_{32}\, qR + 8C_{33} \frac{F_o R^2}{L^2}$$

$$C_4 = C_{41}\, pR^2 + C_{42}\, qR + 8C_{43} \frac{F_o R^2}{L^2}.$$

The constants of integration are obtained from the conditions (4.12) through (4.14). They are given as :

$$D(x) = 0$$

$$E(x) = \frac{\beta}{2a_1a_2}\left(p\ \phi_o + \frac{q}{R} - \frac{8a_4F_o}{L^2}\right)(\phi_o \cos\beta - \sin\phi_o)(L - 2x)$$
$$+ \frac{4F_o}{L^2}(L - 2x)$$

$$A_{11} = A_{12} = A_{13} = 0$$

$$A_{21} = \frac{\phi_o\beta}{4a_1a_2} f_3 + \frac{\phi_o}{a_1}\sin\beta$$

$$A_{22} = \frac{A_{21}}{\phi_o}$$

$$A_{23} = \frac{-a_4\beta}{4a_1a_2} f_3 - \frac{a_3}{a_1}\sin\beta$$

$$B_{11} = \frac{\phi_o\beta}{4a_1a_2}(f_1 - \cos 2\beta - 3) - \frac{\phi_o}{a_1} f_2 + \phi_o \tan\phi_o$$

$$B_{12} = \frac{B_{11}}{\phi_o}$$

$$B_{13} = -\frac{a_4\beta}{4a_1a_2}(f_1 - \cos 2\beta - 3) + \frac{a_3}{a_1} f_2 - \bar{a}\frac{H}{R}\tan\phi_o$$

$$B_{21} = \frac{\phi_o\beta}{4a_1a_2} f_1 - \frac{\phi_o}{a_1}\tan\phi_o\sin\beta + \phi_o\tan\phi_o$$

$$B_{22} = \frac{B_{21}}{\phi_o}$$

$$B_{23} = -\frac{a_4\beta}{4a_1a_2} f_1 + \frac{a_3}{a_1}\sin\beta\tan\phi_o - \bar{a}\frac{H}{R}\tan\phi_o$$

$$C_{11} = C_{12} = C_{13} = 0$$

$$C_{21} = \cos\phi_o + \frac{\phi_o}{2a_1a_2} f_7$$

$$C_{22} = \frac{f_7}{2a_1a_2} - \sin\phi_o$$

$$C_{23} = -\frac{a_4\beta}{2a_1a_2} f_8 + \frac{a_3}{2a_1} f_9 + \bar{a}\frac{H}{R}\sin\phi_o$$

$$C_{31} = \frac{\phi_o\beta}{a_1a_2} f_4 - \frac{\phi_o\beta}{a_1}$$

$$C_{32} = \frac{C_{31}}{\phi_o}$$

$$C_{33} = -\frac{a_4\beta}{a_1a_2} f_4 + \frac{a_3\beta}{a_1}$$

$$C_{41} = \cos\phi_o + \frac{\phi_o\beta}{2a_1a_2} f_5$$

$$C_{42} = \frac{\beta f_5}{2a_1a_2} - \sin\phi_o$$

$$C_{43} = \frac{a_4\beta}{2a_1a_2} f_6 - \frac{a_3}{a_1}\phi_o\beta + \bar{a}\frac{H}{R}\sin\phi_o$$

where f_i, $i = 1,9$ are given by

$$f_1 = \tan\phi_o \left[4\cos\beta\ \mathrm{cosec}\ \phi_o - \sin 2\beta - 2(\phi_o - \beta)\right]$$

$$f_2 = \cos\beta + \sin\beta\tan\phi_o$$

$$f_3 = \sin 2\beta - 2\beta$$

$$f_4 = \sin\beta - \beta\cos\beta$$

$$f_5 = \phi_o^2\cos\beta - 2\phi_o\sin\phi_o - 2\cos\phi_o - 2\cos\beta$$

$$f_6 = \cos\beta\,(\phi_o^2 - 2\phi_o\beta + 2) + 2\phi_o\sin\beta + 2\cos\phi_o$$

$$f_7 = 2\phi_o \cos\beta + \phi_o(\phi_o - \beta)\beta \cos\beta + \beta^2(\sin\phi_o + \sin\beta)$$
$$-2\beta(\cos\phi_o + \phi_o \sin\phi_o) - 2\sin\phi_o + 2\sin\beta$$

$$f_8 = 2(\cos\beta - \cos\phi_o) - (\phi_o - \beta)^2 \cos\beta - 2(\phi_o - \beta)\sin\beta$$

$$f_9 = \beta^2 - 2\phi_o\beta - 2 \; .$$

The stress resultants are summarised after substituting the constants of integration in the modified form as follows :

(i) For $0 \leq \phi \leq \beta$

$$N_x = -\frac{1}{2a_1 R}\left(p\,\phi_o + \frac{q}{R} - \frac{8F_o\, a_3}{L^2}\right)(x)(L - x) \quad \ldots (4.84)$$

$$N_{x\phi} = \frac{\phi}{2a_1}\left(p\,\phi_o + \frac{q}{R} - \frac{8F_o\, a_3}{L^2}\right)(L - 2x) \quad \ldots (4.85)$$

$$N_\phi = pR\left(\frac{\phi_o}{a_1} + B_{11}\cos\phi - \phi\sin\phi\right) + q\left(\frac{1}{a_1} + B_{12}\cos\phi\right)$$
$$+ \frac{8F_o R}{L^2}\left(B_{13}\cos\phi - \frac{a_3}{a_1}\right) \quad \ldots (4.86)$$

$$M_\phi = pR^2\left(B_{11}\cos\phi - \frac{\phi_o\,\phi^2}{2a_1} - \cos\phi - \phi\sin\phi + C_{21}\right)$$
$$+ qR\left(B_{12}\cos\phi + C_{22} - \frac{\phi^2}{2a_1}\right) + \frac{8F_o R^2}{L^2}\left(\frac{a_3\,\phi^2}{2a_1} + B_{13}\cos\phi + C_{23}\right)$$
$$\ldots (4.87)$$

$$Q_\phi = -pR\left(B_{11}\sin\phi + \frac{\phi_o\phi}{a_1} + \phi\cos\phi\right) - q\left(B_{12}\sin\phi + \frac{\phi}{a_1}\right)$$
$$+ \frac{8F_o R}{L^2}\left(\frac{a_3\phi}{a_1} - B_{13}\sin\phi\right) \quad \ldots (4.88)$$

(ii) For $\beta \leq \phi \leq \phi_o$

$$N_x = \frac{\beta}{2a_1 a_2 R} \left(p \ \phi_o + \frac{q}{R} - \frac{8F_o a_4}{L^2}\right) (\cos \beta - \cos \phi) \ (x) \ (L-x) \qquad \ldots (4.89)$$

$$N_{x\phi} = \frac{\beta}{2a_1 a_2} \left(p \ \phi_o + \frac{q}{R} - \frac{8F_o \ a_4}{L^2}\right) \left[(\phi_o - \phi) \cos \beta - \sin \phi_o + \sin\phi \right] (L - 2x) + \frac{4F_o}{L^2} (L - 2x) \qquad \ldots (4.90)$$

$$N_\phi = pR \left[A_{21} \sin \phi + B_{21} \cos \phi - \phi \sin \phi - \frac{\phi_o \ \beta}{a_1 a_2} \left(\cos \beta - \frac{\phi \sin \phi}{2}\right)\right] + q\left[A_{22} \sin \phi + B_{22} \cos \phi - \frac{\beta}{a_1 a_2} \left(\cos \beta - \frac{\phi \sin \phi}{2}\right) \right] + \frac{8F_o R}{L^2} \left[A_{23} \sin \phi + B_{23} \cos \phi + \frac{a_4 \ \beta}{a_1 a_2} \left(\cos \beta - \frac{\phi \sin \phi}{2}\right) \right] \qquad \ldots (4.91)$$

$$M_\phi = pR^2 \left(A_{21} \sin \phi + B_{21} \cos \phi + C_{31} \phi + C_{41} - \cos \phi - \phi \sin \phi - \frac{\phi_o \ \beta \ F_1(\phi)}{2a_1 a_2}\right) + qR \left(A_{22} \sin \phi + B_{22} \cos \phi + C_{32} \phi + C_{42} + \frac{\beta \ F_1(\phi)}{2a_1 a_2}\right) + \frac{8F_o R^2}{L^2} \left(A_{23} \sin \phi + B_{23} \cos \phi + C_{33} \phi + C_{43} - \frac{\beta \ a_4 \ F_1(\phi)}{2a_1 a_2}\right) \qquad \ldots (4.92)$$

$$Q_{\phi} = pR(A_{21} \cos \phi - B_{21} \sin \phi + C_{31} - \phi \cos \phi + \frac{\phi_o \beta \, F_2(\phi)}{2a_1 a_2})$$

$$+ q(A_{22} \cos \phi - B_{22} \sin \phi + C_{32} + \frac{\beta \, F_2(\phi)}{2a_1 a_2})$$

$$+ \frac{8F_o \, R}{L^2} (A_{23} \cos \phi - B_{23} \sin \phi + C_{33} - \frac{\beta \, a_4 \, F_2(\phi)}{2a_1 a_2})$$

$$\ldots (4.93)$$

where

$$F_1(\phi) = \phi^2 \cos \beta + 2 \cos \phi + \phi \sin \phi$$

$$F_2(\phi) = 2 \phi \cos \beta - \sin \phi + \phi \cos \phi$$

From the expressions for N_{ϕ} , M_{ϕ} and Q_{ϕ} it can be found that they are independent of x.

4.6.2 Determination of Limit Load

The limit load p is the smaller of the two values p_{L1} and p_{L2} evaluated from the two independent yield conditions, viz., N_c and $M_{\phi} - {}_{\phi}$ yield criteria respectively. These are determined as follows :

(i) Due to N_c-yield criterion : The maximum value of N_x in compression which occurs at centre of the span is limited to N_c. It is determined from Eq. (4.84). Thus

$$N_{x \ max} = - \frac{1}{8a_1 R} (p \ \phi_o + \frac{q}{R} - \frac{8F_o \ a_3}{L^2}) \ L^2 \ . \qquad \ldots (4.94)$$

The yield condition (2.26) and Eq. (4.94) give

$$p_{L1} = \frac{8F_o a_1 R}{\phi_o L^2} - \frac{q}{\phi_o R} + \frac{8F_o a_3}{\phi_o L^2} . \qquad (4.95)$$

(ii) Due to $N_\phi - M_\phi$ yield criterion : As explained earlier in the Section 4.5.? the failure of the shell occurs in the transverse direction at the crown. Representing the values of N_ϕ and M_ϕ at the crown by $N_{\phi c}$ and $M_{\phi c}$ (they are evaluated using Eqs. (4.86) and (4.87) respectively) one gets

$$N_{\phi c} = pR(\frac{\phi_o}{a_1} + B_{11}) + q(\frac{1}{a_1} + B_{12}) + \frac{8F_o R}{L^2} (B_{13} - \frac{a_3}{a_1}), \quad \ldots (4.96)$$

$$M_{\phi c} = pR^2(B_{11} + C_{21} - 1) + qR (B_{12} + C_{22}) + \frac{8F_o R^2}{L^2} (B_{13} + C_{23}) .$$

$$\ldots (4.97)$$

Substitution of Eqs. (4.96) and (4.97) in the yield condition given by Eq. (2.30) give

$$p_{L2} = \frac{1}{D_1} \Big[b + c \frac{q}{N_o} (B_{12} + \frac{1}{a_1}) - \frac{qR}{M_o} (B_{12} + C_{22}) + \frac{8c F_o R}{N_o L^2} (B_{13} - \frac{a_3}{a_1}) - \frac{8F_o R^2}{M_o L^2} (B_{13} + C_{23}) \Big] , \quad \ldots (4.98)$$

where

$$D_1 = \frac{R^2}{M_o} (B_{11} + C_{21} - 1) - \frac{cR}{N_o} (B_{11} + \frac{\phi_o}{a_1}) .$$

The lower bound load p is the smaller of the two values p_{L1} and p_{L2}.

4.7 Lower Bound Solution 3

In the lower bound solutions 1 and 2, the distribution of N_x in compression zone is taken as uniform across the cross section. In the lower bound solution 3, the distribution of N_x in compression is taken as parabolic which varies with the vertical distance z' from the neutral axis. The maximum value of N_x in compression in the transverse direction occurs at an angle β_1, measured from the corwn. The value of β_1 is kept as a variable for a shell of given geometry and its value is evaluated such that it gives an optimal value for the ultimate load. The variation of N_x in the tension zone is kept as linear, same as in the lower bound solution 2, which varies with the vertical distance z' below the neutral axis. This is shown in Fig. 4.2d.

The distribution of longitudinal tension in the edge beam is taken as trapezoidal as indicated in Fig. 4.1c. The magnitude of the force per unit depth at top of the edge beam is n times that of the one at the bottom.

The reasons for choosing this kind of distribution for N_x are explained earlier in Section 3.6, while developing lower bound solutions for shells with free longitudinal edges.

4.7.1. Stress resultants

Evaluation of N_x :

N_x is evaluated as follows :

N_q and N_p represent the maximum values of N_x in the compression and tension zones respectively as shown in Fig. 4.2a. The parabolic variation in the compression zone and the linear variation in the tension zone are expressed as

$$N_x = -\frac{1}{z_1'^2} N_q \, z'(2z_1' - z') \qquad \text{for } 0 \leq \phi \leq \beta \, , \qquad \ldots (4.99)$$

and

$$N_x = \frac{N_p \, z'}{h_2} \qquad \text{for } \beta \leq \phi \leq \phi_o , \qquad \ldots (4.100)$$

where z_1' is the vertical distance from the neutral axis to the point of maximum compression and h_2 is the depth of the compression zone. Thus (Fig. 4.2a)

$$z' = R(\cos\phi - \cos\beta) \qquad \text{for } 0 \leq \phi \leq \beta \, ,$$

$$z' = R(\cos\beta - \cos\phi) \qquad \text{for } \beta \leq \phi \leq \phi_o ,$$

$$z_1' = R(\cos\beta_1 - \cos\beta) ,$$

$$h_2 = R(\cos\beta - \cos\phi_o) .$$

Substituting these values in Eqs. (4.99) and (4.100), N_x is expressed as

$$N_x = -\frac{N_q}{(\cos\beta_1 - \cos\beta)^2} (2\cos\beta_1 - \cos\beta - \cos\phi)(\cos\phi - \cos\beta)$$

$$\text{for } 0 \leq \phi \leq \beta \qquad \ldots (4.101)$$

and

$$N_x = \frac{N_p(\cos\beta - \cos\phi)}{(\cos\beta - \cos\phi_o)} \qquad \text{for } \beta \leq \phi \leq \phi_o \, . \qquad \ldots (4.102)$$

where

$$a_1 = \frac{[(\phi_0 - \beta)\cos\beta - \sin\phi_0 + \sin\beta]}{[2\cos\beta_1(\sin\beta - \beta\cos\beta) + \frac{\beta}{2}\cos 2\beta - \frac{1}{4}\sin 2\beta]} .$$

The moment of resistance M_r at any section x is found by taking moments about the neutral axis. Thus

$$M_r = 2\int_0^{\beta} N_x R \, d\phi \, z' + 2\int_{\beta}^{\phi_0} N_x R \, d\phi \, z' + 2 F_x [R(\cos\beta - \cos\phi_0) + \bar{a} H] . \qquad \ldots (4.108)$$

After substituting Eqs. (4.101), (4.102) and (4.107)

$$M_r = \frac{2 a_2 N_q R^2}{(\cos\beta_1 - \cos\beta)^2} + 2 a_3 F_x R , \qquad \ldots (4.109)$$

where

$$a_2 = \cos\beta_1 [\beta(2 + \cos 2\beta) - \frac{3}{2}\sin 2\beta] + \frac{7}{12}\cos\beta \sin 2\beta - \frac{2}{3}\sin\beta - \frac{\beta}{2}\cos\beta\cos 2\beta + \frac{1}{4a_1}[2(\phi_0 - \beta)(2 + \cos 2\beta) + \sin 2\phi_0 - 3\sin\phi_0\cos\beta + 3\sin 2\beta],$$

$$a_3 = \cos\beta - \cos\phi_0 + \bar{a}\frac{H}{R} - \frac{a_4}{4}[2(\phi_0 - \beta)(2 + \cos 2\beta) + \sin 2\phi_0 - 8\sin\phi_0\cos\beta + 3\sin 2\beta]$$

The external bending moment M at any section x is

$$M = (pr\,\phi_0 + q)(x)(L - x) . \qquad \ldots (4.110)$$

Equating Eqs. (4.109) and (4.110) and substituting in Eq. (4.1) for F_x one gets

$$N_q = \frac{1}{2a_2 R}\left[p\ \phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2}\right] (\cos\beta_1 - \cos\beta)^2 (x)(L - x). \qquad \ldots (4.111)$$

Using Eqs. (4.101), (4.102), (4.107) and (4.111) N_x is expressed as

$$N_x = -\frac{1}{2a_2 R}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2}\right)(2\cos\beta_1 - \cos\beta - \cos\phi)$$

$$(\cos\phi - \cos\beta)\ (x)\ (L - x) \quad \text{for } 0 \leq \phi \leq \beta, \qquad \ldots (4.112)$$

$$N_x = \frac{1}{R}\left[\frac{\left(p\ \phi_o + \frac{q}{R} - \frac{8a_3 F_o}{L^2}\right)}{2a_1 a_2} - \frac{4a_4 F_o}{L^2}\right] (\cos\beta - \cos\phi)(x)(L-x)$$

$$\text{for } \beta \leq \phi \leq \phi_o, \qquad \ldots (4.113)$$

where

$$a_4 = \frac{1}{\left[(\phi_o - \beta)\cos\beta - \sin\phi_o + \sin\beta\right]}.$$

<u>Evaluation of $N_{x\phi}$</u> :

$N_{x\phi}$ is determined as follows :

Eq. (2.15) gives

$$\frac{\partial N_{x\phi}}{\partial \phi} = -R\frac{\partial N_x}{\partial x}. \qquad \ldots (4.114)$$

(i) For $0 \leq \phi \leq \beta$

Substitution of Eq. (4.112) in Eq. (4.114) yields

$$\frac{\partial N_{x\phi}}{\partial \phi} = \frac{1}{2a_2} (p\ \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2}) (2 \cos \beta_1 - \cos \beta - \cos \phi)$$

$$(\cos \phi - \cos \beta) (L - 2x), \qquad \ldots (4.115)$$

which on integration with respect to ϕ results in

$$N_{x\phi} = \frac{1}{2a_2} (p\ \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2}) (2 \cos \beta_1 (\sin \phi - \phi \cos \beta)$$

$$+ \frac{\phi}{2} \cos 2\beta - \frac{\sin 2\phi}{4}) (L - 2x) + D(x), \qquad \ldots (4.116)$$

where D(x) is an arbitrary function of x only.

(ii) For $\beta \leq \phi \leq \phi_o$:

From Eqs. (4.113) and (4.114) it is obtained

$$\frac{\partial N_{x\phi}}{\partial \phi} = - \left[\frac{(p\ \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2})}{2a_1a_2} - \frac{4a_4F_o}{L^2} \right] (\cos \beta - \cos \phi)(L-2x).$$

$$\ldots (4.117)$$

On integration with respect to   it yields

$$N_{x\phi} = - \left[\frac{(p\ \phi_o + \frac{q}{R} - \frac{8a_3\ F_o}{L^2})}{2a_1a_2} - \frac{4a_4F_o}{L^2} \right] (\phi \cos \beta - \sin \phi)(L - 2x)$$

$$+ E(x) , \qquad \ldots (4.118)$$

where E(x) is an arbitrary function of x only.

Evaluation of N_ϕ :

N_ϕ is determined as described below :

Eq. (2.16) is rewritten as

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = R^2 \frac{\partial^2 N_x}{\partial x^2} - 2pR \cos \phi \; . \qquad \ldots (4.119)$$

(i) For $0 \leq \phi \leq \beta$

Using Eq. (4.112) for N_x, Eq. (4.119) results in

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = \frac{R}{a_2} \left(p\ \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2}\right) (2 \cos \beta_1 - \cos \beta - \cos \phi)$$

$$(\cos \phi - \cos \beta) - 2pR \cos \phi \; . \qquad \ldots (4.120)$$

The differential equation has the solution of the form

$$N_\phi = A_1 \sin \phi + B_1 \cos \phi + \frac{R}{a_2} \left(p\ \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2}\right)(\phi \sin \phi \cos \beta_1$$

$$- 2 \cos \beta \cos \beta_1 + \frac{\cos 2\beta}{2} + \frac{\cos 2\phi}{6}) - pR\ \phi \sin \phi \, , \qquad \ldots (4.121)$$

where A_1 and B_1 are constants of integration. On differentiation the above equation gives

$$\frac{\partial N_\phi}{\partial \phi} = A_1 \cos \phi - B_1 \sin \phi + \frac{R}{a_2} \left(p\ \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2}\right)$$

$$(\cos \beta_1(\sin \phi + \phi \cos \phi) - \frac{\sin 2\phi}{3}) - pR(\sin \phi + \phi \cos \phi) . \qquad \ldots (4.122)$$

(ii) For $\beta \leq \phi \leq \phi_o$:

Substitution of Eq. (4.113) in Eq. (4.119) yields

$$N_\phi + \frac{\partial^2 N_\phi}{\partial \phi^2} = - R \left[\frac{(p\ \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2})}{a_1a_2} - \frac{4a_4F_o}{L^2} \right] (\cos\beta - \cos\phi)$$

$$- 2\ pR \cos\ \phi . \qquad \ldots (4.123)$$

The solution of the differential equation is

$$N_\phi = A_2 \sin\phi + B_2 \cos\phi - R \left[\frac{(p\ \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2})}{a_1a_2} - \frac{4a_4F_o}{L^2} \right]$$

$$(\cos\beta - \frac{\phi \sin\phi}{2}) - pR\ \phi \sin\phi , \qquad \ldots (4.124)$$

where A_2 and B_2 are constants of integration. On differentiation

$$\frac{\partial N_\phi}{\partial \phi} = A_2 \cos\phi - B_2 \sin\phi + \frac{R}{2} \left[\frac{(p\ \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2})}{a_1a_2} - \frac{4a_4F_o}{L^2} \right]$$

$$(\phi \cos\phi + \sin\phi) - pR\ (\phi\cos\phi + \sin\phi). \qquad \ldots (4.125)$$

Evaluation of M_ϕ :

M_ϕ is evaluated as follows :

From Eq. (2.17)

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = - N_\phi\ R - pR^2 \cos\phi . \qquad \ldots (4.126)$$

(i) For $0 \leq \phi \leq \beta$

Eqs. (4.121) and (4.126) yield

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = -A_1 R \sin \phi - B_1 R \cos \phi - \frac{R^2}{a_2} \left(p \phi_o + \frac{q}{R} - \frac{8 a_3 F_o}{L^2}\right)$$

$$\left(\phi \sin \phi \cos \beta_1 - 2 \cos \beta \cos \beta_1 + \frac{\cos 2\beta}{2} + \frac{\cos 2\phi}{6}\right)$$

$$+ pR^2 (\phi \sin \phi - \cos \phi). \qquad (4.127)$$

Integrating once

$$\frac{\partial M_\phi}{\partial \phi} = A_1 R \cos \phi - B_1 R \sin \phi - \frac{R^2}{a_2} \left(p \phi_o + \frac{q}{R} - \frac{8 a_3 F_o}{L^2}\right)$$

$$\left(\cos \beta_1 (\sin \phi - \phi \cos \phi) - 2\phi \cos \beta_1 \cos \beta + \frac{\phi}{2} \cos 2\beta\right.$$

$$\left. + \frac{\sin 2\phi}{12}\right) - pR^2 \phi \cos \phi + C_1 , \qquad \ldots (4.128)$$

where C_1 is a constant of integration. Integrating once again

$$M_\phi = A_1 R \sin \phi + B_1 R \cos \phi + \frac{R^2}{a_2} \left(p \phi_o + \frac{q}{R} - \frac{8 a_3 F_o}{L^2}\right)$$

$$\left(\cos \beta_1 (2 \cos \phi + \phi \sin \phi) + \phi^2 \cos \beta_1 \cos \beta\right.$$

$$\left. - \frac{\phi^2}{4} \cos 2\beta + \frac{\cos 2\phi}{24}\right) - pR^2 (\cos \phi + \phi \sin \phi) + C_1 \phi$$

$$+ C_2 , \qquad \ldots (4.129)$$

where C_2 is a constant of integration.

(ii) For $\beta \leq \phi \leq \phi_o$

Substituting Eq. (4.124) in Eq. (4.126)

$$\frac{\partial^2 M_\phi}{\partial \phi^2} = - A_2 R \sin \phi - B_2 R \cos \phi + R^2 \left[\frac{(p \ \phi_o + \frac{q}{R} - \frac{8a_3 F_o}{L^2})}{a_1 a_2} - \frac{4a_4 F_o}{L^2} \right] (\cos \beta - \frac{\phi}{2} \sin \phi) + pR^2 (\phi \sin \phi - \cos \phi) . \qquad \ldots (4.130)$$

Integrating twice

$$\frac{\partial M_\phi}{\partial \phi} = A_2 R \cos \phi - B_2 R \sin \phi + R^2 \left[\frac{(p \ \phi_o + \frac{q}{R} - \frac{8a_3 F_o}{L^2})}{a_1 a_2} - \frac{4a_4 F_o}{L^2} \right] (\phi \cos \beta - \frac{1}{2} \sin \phi + \frac{\phi}{2} \cos \phi) - pR^2 \phi \cos \phi + C_3 , \qquad \ldots (4.131)$$

$$M_\phi = A_2 R \sin \phi + B_2 R \cos \phi + R^2 \left[\frac{(p \ \phi_o + \frac{q}{R} - \frac{8a_3 F_o}{L^2})}{a_1 a_2} - \frac{4a_4 F_o}{L^2} \right] (\frac{\phi^2}{2} \cos \beta + \cos \phi + \frac{\phi}{2} \sin \phi) - pR^2 (\cos \phi + \phi \sin \phi) + C_3 \phi + C_4 , \qquad \ldots (4.132)$$

where C_3 and C_4 are constants of integration.

Evaluation of Q_ϕ :

Q_ϕ is determined as described :

Eq. (2.18) may be written as

$$Q_\phi = \frac{1}{R}\frac{\partial M_\phi}{\partial \phi} \; . \qquad \ldots (4.133)$$

Substitution of Eqs. (4.128) and (4.131) in Eq. (4.133) gives

$$Q_\phi = A_1 \cos\phi - B_1 \sin\phi - \frac{R}{a_2}\left(p\ \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2}\right)\left(\cos\beta_1(\sin\phi - \phi\cos\phi) - 2\phi\cos\beta_1\cos\beta + \frac{\phi}{2}\cos 2\beta + \frac{\sin 2\phi}{12}\right) - pR\,\phi\cos\phi + \frac{C_1}{R} , \qquad \ldots (4.134)$$

and

$$Q_\phi = A_2 \cos\phi - B_2 \sin\phi + R\left[\frac{(p\ \phi_o + \frac{q}{R} - 8a_3F_o)}{a_1 a_2} - \frac{4a_4F_o}{L^2}\right](\phi\cos\beta - \frac{1}{2}\sin\phi + \frac{\phi}{2}\cos\phi) - pR\,\phi\cos\phi + \frac{C_3}{R} , \qquad \ldots (4.135)$$

respectively.

The constants of integration are modified as indicated :

$$A_1 = A_{11}\, pR + A_{12}\, q + 8A_{13}\frac{F_oR}{L^2} ,$$

$$B_1 = B_{11}\, pR + B_{12}\, q + 8B_{13}\frac{F_oR}{L^2} ,$$

$$A_2 = A_{21}\, pR + A_{22}\, q + 8A_{23}\frac{F_oR}{L^2} ,$$

$$B_2 = B_{21}\, pR + B_{22}\, q + 8B_{23}\frac{F_oR}{L^2} ,$$

$$C_1 = C_{11}\, pR^2 + C_{12}\, qR + 8C_{13}\, \frac{F_o R^2}{L^2},$$

$$C_2 = C_{21}\, pR^2 + C_{22}\, qR + 8C_{23}\, \frac{F_o R^2}{L^2},$$

$$C_3 = C_{31}\, pR^2 + C_{32}\, qR + 8C_{33}\, \frac{F_o R^2}{L^2},$$

$$C_4 = C_{41}\, pR^2 + C_{42}\, qR + 8C_{43}\, \frac{F_o R^2}{L^2}.$$

The constants of integration are determined from the conditions given by Eqs. (4.12) through (4.14). Thus

$$D(x) = 0$$

$$E(x) = \left[\frac{(p\, \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2})}{2a_1a_2} - \frac{4a_4F_o}{L^2} \right] (\phi_o \cos \beta - \sin \phi_o)$$

$$(L - 2x) + \frac{4F_o}{L^2}(L - 2x)$$

$$A_{11} = A_{12} = A_{13} = 0$$

$$A_{21} = \frac{\phi_o\, f_1}{4a_1a_2} - \frac{\phi_o\, f_2}{2a_2}$$

$$A_{22} = \frac{A_{21}}{\phi_o}$$

$$A_{23} = -\left(\frac{a_3}{4a_1a_2} + \frac{a_4}{8}\right) f_1 + \frac{a_3}{2a_2}\, f_2$$

$$B_{11} = \phi_o \tan \phi_o \left(1 + \frac{f_2}{2a_2} - \frac{f_3}{4a_1a_2}\right) + \frac{\phi_o\, f_4}{2a_2} - \frac{\phi_o\, f_5}{4a_1a_2}$$

$$B_{12} = \frac{B_{11}}{\phi_o}$$

$$B_{13} = \left(\frac{a_3}{4a_1a_2} + \frac{a_4}{8}\right)(f_5 + f_3 \tan \phi_o) - \frac{a_3}{2a_2}(f_4+f_2 \tan \phi_o)$$

$$- \bar{a} \frac{H}{R} \tan \phi_o$$

$$B_{21} = \phi_o \tan \phi_o \left(1 + \frac{f_2}{2a_2} - \frac{f_3}{4a_1a_2}\right)$$

$$B_{22} = \frac{B_{21}}{\phi_o}$$

$$B_{23} = - \frac{a_3 \ f_2}{2a_2} \tan \phi_o + \left(\frac{a_3}{4a_1a_2} + \frac{a_4}{8}\right) f_3 \tan \phi_o - \bar{a} \frac{H}{R} \tan \phi_o$$

$$C_{11} = C_{12} = C_{13} = 0$$

$$C_{21} = \cos \phi_o + \frac{\phi_o \ f_9}{2a_1a_2} + \frac{\phi_o(\phi_o - \beta) \ f_6}{4a_2} - \frac{\phi_o \ f_{10}}{8a_2}$$

$$C_{22} = \frac{f_9}{2a_1a_2} + \frac{(\phi_o-\beta) \ f_6}{4a_2} - \frac{f_{10}}{8a_2} - \sin \phi_o$$

$$C_{23} = -\left(\frac{a_3}{2a_1a_2} + \frac{a_4}{4}\right) f_9 - \frac{a_3}{4a_2} (\phi_o-\beta)f_6 + \frac{a_3f_{10}}{8a_2} + \bar{a} \frac{H}{R} \sin \phi_o$$

$$C_{31} = \frac{\phi_o \ f_7}{a_1a_2} - \frac{\phi_o \ f_6}{4a_2}$$

$$C_{32} = \frac{C_{31}}{\phi_o}$$

$$C_{33} = \frac{a_3 \ f_6}{4a_2} - \left(\frac{a_3}{a_1a_2} + \frac{a_4}{2}\right) f_7$$

$$C_{41} = \cos \phi_o - \frac{\phi_o \ f_8}{2a_1a_2} + \frac{\phi_o^2 \ f_6}{4a_2}$$

$$C_{42} = \frac{\phi_o f_6}{4a_2} - \frac{f_8}{2a_1a_2} - \sin\phi_o$$

$$C_{43} = \left(\frac{a_3}{2a_1a_2} + \frac{a_4}{4}\right) f_8 - \frac{a_3\phi_o f_6}{4a_2} + \bar{a}\frac{H}{R}\sin\phi_o$$

where f_i, $i = 1, 10$ are given by

$$f_1 = \sin 2\beta - 2\beta$$

$$f_2 = \cos\beta_1 (\sin 2\beta - 2\beta) + \frac{2}{3}\sin\beta(1 - \cos 2\beta)$$

$$f_3 = \sin 2\beta - 4\cos\beta \ \mathrm{cosec}\,\phi_o + 2(\phi_o - \beta)$$

$$f_4 = 3\cos\beta_1 + (\cos\beta_1 - \frac{2}{3}\cos\beta)\cos 2\beta - \frac{2}{3}\cos\beta$$

$$f_5 = 3 + \cos 2\beta$$

$$f_6 = 2\beta\cos 2\beta + 8\cos\beta_1(\sin\beta - \beta\cos\beta) - \sin 2\beta$$

$$f_7 = \sin\beta - \beta\cos\beta$$

$$f_8 = 2\cos\beta + 2\cos\phi_o + 2\phi_o\sin\beta + \phi_o(\phi_o - 2\beta)\cos\beta$$

$$f_9 = 2\cos\beta - 2\cos\phi_o - 2(\phi_o - \beta)\sin\beta - (\phi_o - \beta)^2\cos\beta$$

$$f_{10} = 3(4 + \beta^2)\cos\beta\cos\beta_1 - (5 + 2\beta^2)\cos 2\beta$$

The stress resultants are summarised after substituting the constants of integration in the modified form as follows :

(i) For $0 \leq \phi \leq \beta$

$$N_x = -\frac{1}{2a_2 L}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2}\right) F_1(\phi)\ (x)\ (L-2x) \qquad \ldots (4.136)$$

$$N_{x\phi} = \frac{1}{2a_2}\left(p\ \phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2}\right) F_2(\phi)\ (L - 2x) \qquad \ldots (4.137)$$

$$N_\phi = pR\left(B_{11} \cos\phi - \phi \sin\phi + \phi_o \frac{F_3(\phi)}{a_2}\right) + q\left(B_{12} \cos\phi + \frac{F_3(\phi)}{a_2}\right) + \frac{8F_o R}{L^2}\left(B_{13} \cos\phi - \frac{a_3}{a_2} F_3(\phi)\right) \qquad \ldots (4.138)$$

$$M_\phi = pR^2\left(B_{11} \cos\phi + \frac{\phi_o}{a_2} F_4(\phi) + C_{21} - \cos\phi - \phi \sin\phi\right) + qR\left(B_{12} \cos\phi + \frac{F_4(\phi)}{a_2} + C_{22}\right) + \frac{8F_o R^2}{L^2}\left(B_{13} \cos\phi - \frac{a_3 F_4(\phi)}{a_2} + C_{23}\right) \qquad \ldots (4.139)$$

$$Q_\phi = pR\left(\frac{\phi_o}{a_2} F_9(\phi) - \phi \cos\phi - B_{11} \sin\phi\right) + q\left(\frac{F_9(\phi)}{a_2} - B_{12} \sin\phi\right) - \frac{8F_o R}{L^2}\left(\frac{a_3\ F_9(\phi)}{a_2} + B_{13} \sin\phi\right) \qquad \ldots (4.140)$$

(ii) For $\beta \leq \phi \leq \phi_o$

$$N_x = \frac{1}{R}\left[\frac{\left(p\ \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2}\right)}{2a_1a_2} - \frac{4a_4F_o}{L^2}\right] F_5(\phi)(x)(L-x) \ldots (4.141)$$

$$N_{x\phi} = \left[\frac{\left(p\ \phi_o + \frac{q}{R} - \frac{8a_3F_o}{L^2}\right)}{2a_1a_2} - \frac{4a_4F_o}{L^2}\right] F_6(\phi)(L-2x) + \frac{4F_o}{L^2}(L-2x) \qquad \ldots (4.142)$$

$$N_\phi = pR(A_{21} \sin \phi + B_{21} \cos \phi - \phi \sin \phi - \frac{\phi_o}{a_1 a_2} F_7(\phi))$$

$$+ q(A_{22} \sin \phi + B_{22} \cos \phi - \frac{F_7(\phi)}{a_1 a_2})$$

$$+ \frac{8F_o R}{L^2} \left[A_{23} \sin \phi + B_{23} \cos \phi + (\frac{a_3}{a_1 a_2} + \frac{a_4}{2}) F_7(\phi) \right]$$

... (4.143)

$$M_\phi = pR^2(A_{21} \sin \phi + B_{21} \cos \phi - \cos \phi - \phi \sin \phi + \frac{\phi_o}{a_1 a_2} F_8(\phi)$$

$$+ C_{31} \phi + C_{41}) + qR(A_{22} \sin \phi + B_{22} \cos \phi + \frac{F_8(\phi)}{a_1 a_2}$$

$$+ C_{32} \phi + C_{42}) + \frac{8F_o R^2}{L^2} \left[A_{23} \sin \phi + B_{23} \cos \phi \right.$$

$$\left. - (\frac{a_3}{a_1 a_2} + \frac{a_4}{2}) F_8(\phi) + C_{33} \phi + C_{43} \right]$$

... (4.144)

$$Q_\phi = pR(A_{21} \cos \phi - B_{21} \sin \phi - \phi \cos \phi + \frac{\phi_o}{a_1 a_2} F_{10}(\phi) + C_{31})$$

$$+ q(A_{22} \cos \phi - B_{22} \sin \phi + \frac{F_{10}(\phi)}{a_1 a_2} + C_{32})$$

$$+ \frac{8F_o R}{L^2} \left[A_{23} \cos \phi - B_{23} \sin \phi - (\frac{a_3}{a_1 a_2} + \frac{a_4}{2}) F_{10}(\phi) + C_{33} \right]$$

... (4.145)

where

$$F_1(\phi) = (2 \cos \beta_1 - \cos \beta - \cos \phi)(\cos \phi - \cos \beta)$$

$$F_2(\phi) = 2\cos\beta_1(\sin\phi - \phi\cos\beta) + \frac{\phi}{2}\cos 2\beta - \frac{\sin 2\phi}{4}$$

$$F_3(\phi) = \phi\sin\phi\cos\beta_1 - 2\cos\beta\cos\beta_1 + \frac{\cos 2\beta}{2} + \frac{\cos 2\phi}{6}$$

$$F_4(\phi) = \cos\beta_1(2\cos\phi + \phi\sin\phi) + \phi^2\cos\beta_1\cos\beta - \frac{\phi^2}{4}\cos 2\beta + \frac{\cos 2\phi}{24}$$

$$F_5(\phi) = \cos\beta - \cos\phi$$

$$F_6(\phi) = (\phi_o - \phi)\cos\beta - \sin\phi_o + \sin\phi$$

$$F_7(\phi) = \cos\beta - \frac{\phi\sin\phi}{2}$$

$$F_8(\phi) = \frac{\phi^2}{2}\cos\beta + \cos\phi + \frac{\phi\sin\phi}{2}$$

$$F_9(\phi) = \cos\beta_1(\phi\cos\phi - \sin\phi) + 2\phi\cos\beta_1\cos\beta - \frac{\phi}{2}\cos 2\beta - \frac{\sin 2\phi}{12}$$

$$F_{10}(\phi) = \phi\cos\beta - \frac{\sin\phi}{2} + \frac{\phi\cos\phi}{2}$$

4.7.2 Determination of Limit Load

The limit loads which are determined using the N_c and $M_\phi - N_\phi$ yield criteria are denoted by p_{L1} and p_{L2} respectively. The actual lower bound p is the smaller of the two values. This is determined as follows :

(i) Due to N_c-yield criterion : The maximum value of N_x in compression occurs at $x = \frac{L}{2}$ in the longitudinal direction and

at $\phi = \beta_1$ in the transverse direction. Therefore from Eq. (4.136) it is obtained

$$N_{x\,max} = -\frac{L^2}{8a_2R}\left(p\,\phi_o + \frac{q}{R} - \frac{8F_o a_3}{L^2}\right)(\cos\beta_1 - \cos\beta)^2. \quad \ldots (4.146)$$

The yield condition given by (2.26) and Eq. (4.146) give

$$p_{L1} = \frac{8a_2 F_c R}{L^2\phi_o(\cos\beta_1 - \cos\beta)^2} - \frac{q}{\phi_o R} + \frac{8F_o a_3}{\phi_o L^2}. \quad \ldots (4.147)$$

(ii) Due to M_ϕ-N_ϕ yield criterion : As explained earlier in the Section 4.5.2 the failure of the shell occurs in the transverse direction at the crown. Representing the values of N_ϕ and M_ϕ at the crown by $N_{\phi c}$ and $M_{\phi c}$ respectively, they are evaluated using Eqs. (4.138) and (4.139). Thus

$$N_{\phi c} = pR\left(B_{11} + \frac{\phi_o}{a_2}X_1\right) + q\left(B_{12} + \frac{X_1}{a_2}\right) + \frac{8F_oR}{L^2}\left(B_{13} - \frac{a_3}{a_2}X_1\right), \quad \ldots (4.148)$$

and

$$M_{\phi c} = pR^2\left(B_{11} + \frac{\phi_o}{a_2}X_2 + C_{21} - 1\right) + qR\left(B_{12} + \frac{X_2}{a_2} + C_{22}\right)$$

$$+ \frac{8F_oR^2}{L^2}\left(B_{13} + C_{23} - \frac{a_3}{a_2}X_2\right), \quad \ldots (4.149)$$

where

$$X_1 = \frac{1}{6} + \frac{\cos 2\beta}{2} - 2\cos\beta\,\cos\beta_1,$$

$$X_2 = \frac{1}{24} + 2\cos\beta_1.$$

Substitution of $N_{\phi c}$ and $M_{\phi c}$ in the yield condition given by Eq. (2.30) give

$$p_{L2} = \frac{1}{D_1}\Big[b + c\,\frac{q}{N_o}\,(B_{12} + \frac{X_1}{a_2}) - \frac{qR}{M_o}\,(B_{12} + \frac{X_2}{a_2} + C_{22})$$

$$+ \frac{8c\,F_oR}{L^2}\,(B_{13} - \frac{a_3}{a_2}\,X_1) - \frac{8F_oR^2}{M_o\,L^2}\,(B_{13} + C_{23} - \frac{a_3}{a_2}\,X_2)\Big], \quad \ldots (4.150)$$

where

$$D_1 = \frac{R^2}{M_o}\,(B_{11} + \frac{\phi_o}{a_2}\,X_2 + C_{21} - 1) - c\,\frac{R}{N_o}\,(B_{11} + \frac{\phi_o}{a_2}\,X_1).$$

The limit load p is the smaller of the two values p_{L1} and p_{L2}.

4.8 Analysis of Shells

Two shells with different geometric parameters are analysed by both elastic (26) and the proposed lower bound solutions. The ultimate load and the stress resultants are computed at collapse. In the limit analysis the amount of transverse reinforcement is taken such that $\mu = 0.236$. σ_c and σ_{sy} are equal to 150 kg/cm^2 and 2600 kg/cm^2 respectively.

For estimating the collapse load for a given thickness of shell, 't' will be the total thickness of the shell while using the N_c-yield criterion and the effective thickness while using the $M_\phi - N_\phi$ yield criterion.

It is found from the analysis that the limit load increases as η decreases, while keeping the other parameters as constant. Therefore η is taken as equal to 0.2 in the present analysis.

4.8.1 Numerical Example 1 (Long Shell) :

The geometric parameters of the shell 1 are :

$\phi_0 = 30^\circ$, $\beta = \frac{2}{3}\phi_0$, $\beta_1 = 0.1417$, $\frac{L}{R} = 3.333$, $\frac{R}{t} = 100$, $R = 8.0$ m.

The dimensions of the edge beams are : $2W = 25.0$ cms, $2H = 150.0$ cms. The factor η is taken as 0.2. The longitudinal steel in the edge beam is taken as 77.0 cm^2 same as in the elastic design*. Elastic analysis is done for a working load of intensity 300 kg/m^2. This shell fails due to N_c-yield criterion. The solution 2 gives the best lower bound which is equal to 1024.0 kg/m^2 and hence the load factor is 3.41. The values of the reduced stress resultants at critical sections at collapse are tabulated in Table 4.1. The reduced stress resultants are plotted at critical sections for both the elastic and the limit analysis solutions for a comparative study. The variations of the stress resultants with $\frac{y_1}{h}$ are shown graphically in Fig. 4.3.

*The method of design procedure is as given in the reference 26.

Table 4.1 : Reduced stress resultants at critical sections for shell 1 (Long Shell).

ϕ	$\frac{N_x}{N_o}$ $(x = L/2)$	$\frac{N_\phi}{N_o}$	$\frac{M_\phi}{M_o}$	$\frac{N_x}{S_o}$ $(x = 0)$
0^o	-1.0	-0.0773	0.2272	0.0000
5^o	-1.0	-0.0748	0.2404	1.0486
10^o	-1.0	-0.0673	0.2732	2.0973
15^o	-1.0	-0.0547	0.3050	3.1459
20^o	0.0011	-0.0373	0.3009	4.1930
25^o	0.7800	-0.0185	0.2155	3.7978
30^o	1.7175	-0.0048	0.0000	2.5050

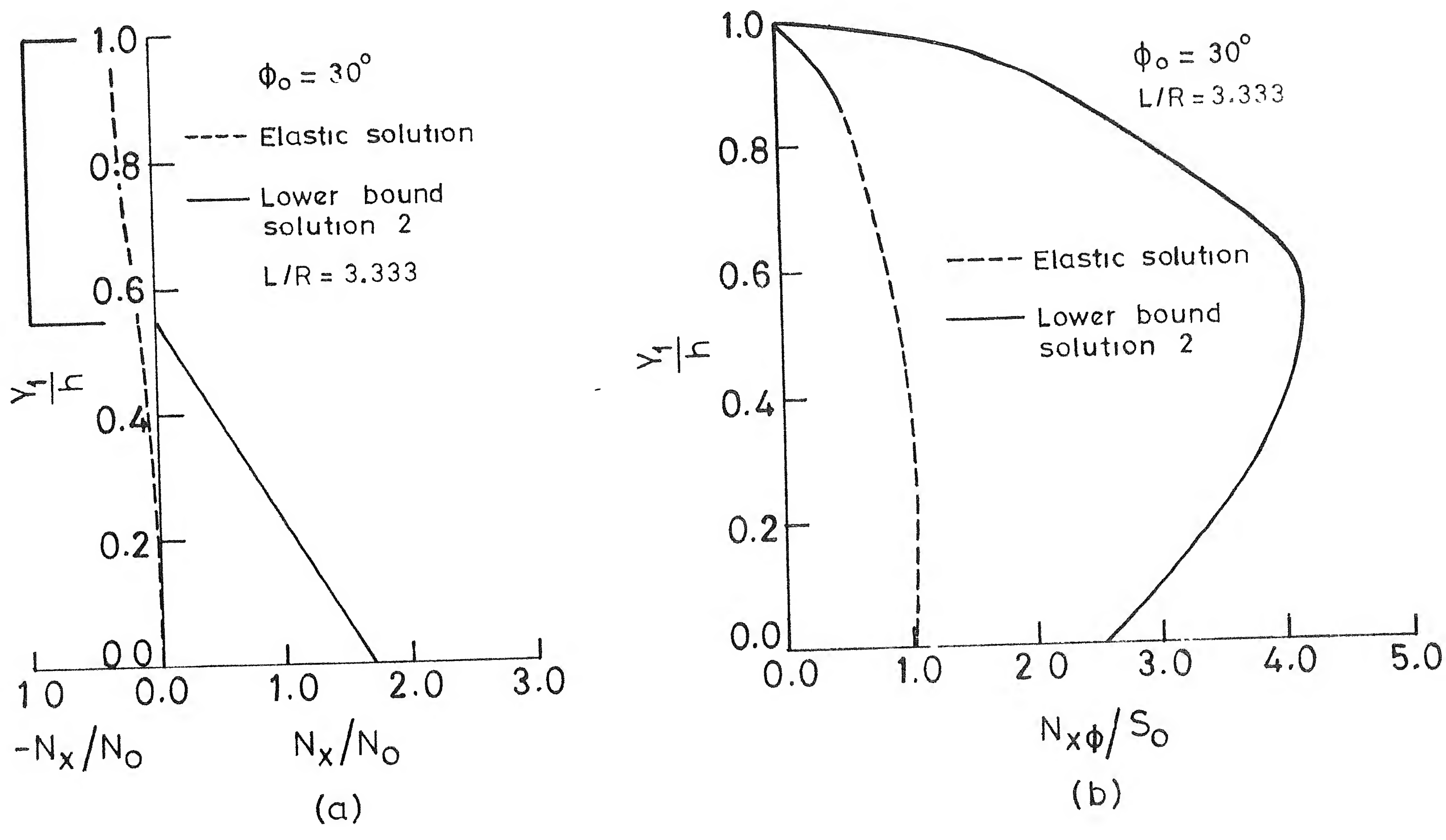


FIG. 4.3 DISTRIBUTION OF STRESS RESULTANTS FOR SHELL 1

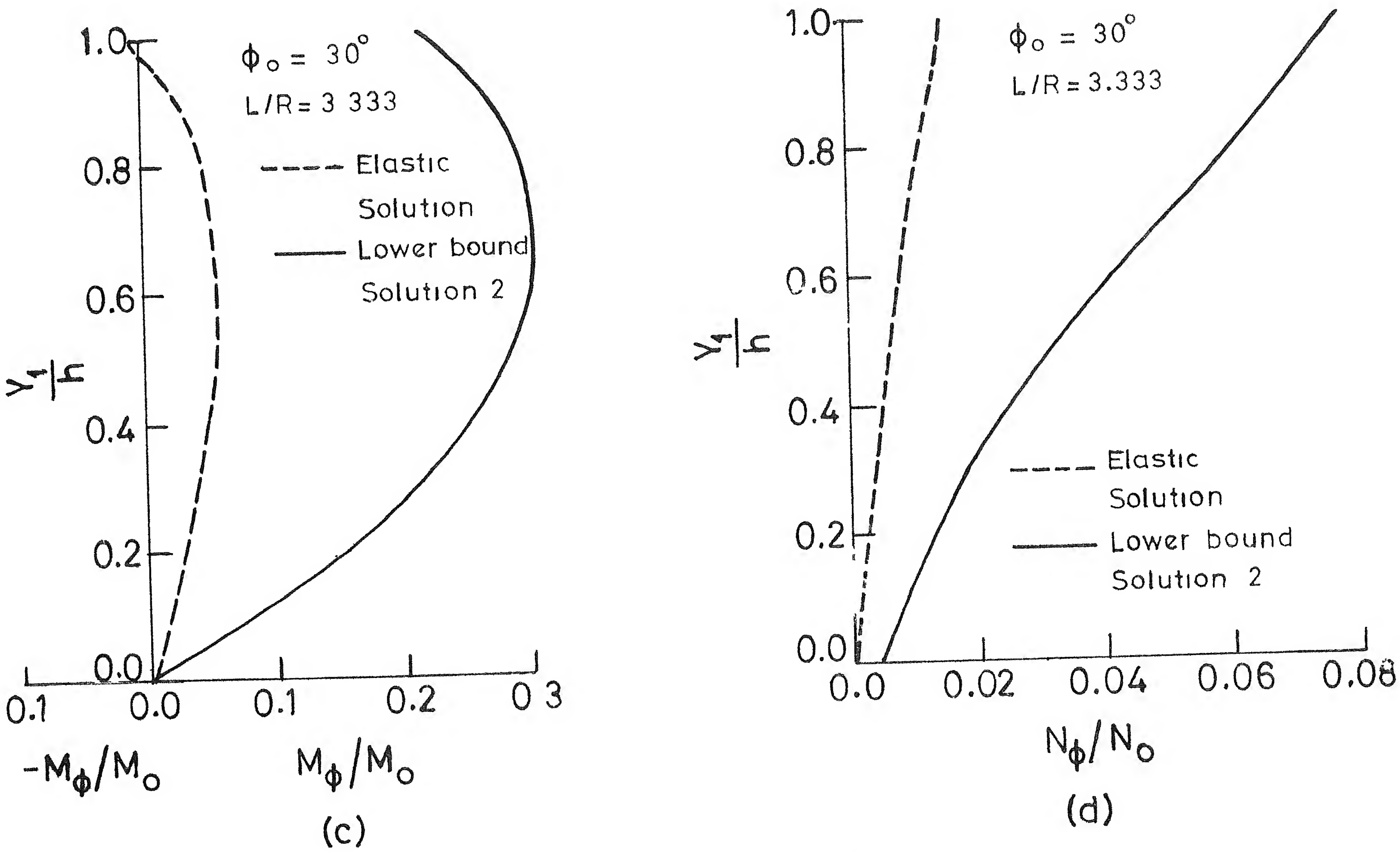


FIG 4.3 DISTRIBUTION OF STRESS RESULTANTS FOR SHELL 1

4.8.2 Numerical Example 2 (Medium Shell) :

The geometric parameters of the shell 2 are :
$\phi_o = 45^o$, $\beta = \frac{2}{3}\phi_o$, $\frac{L}{R} = 2.5$, $\frac{R}{t} = 100$, $R = 8.0$ m. The dimensions of the edge beam are : 2W = 30.0 cms and 2H = 67.50 cms. The factor η is taken as 0.2. The longitudinal steel in the edge beam is taken as 58.0 cm^2 same as in the elastic design. Elastic design is done for a working load of 400 kg/m^2. The solution 3 gives the best lower bound which is equal to 1920 kg/m^2 giving a load factor of 4.8. The shell fails due to $M_\phi - N_\phi$ criterion. The optimal value of β_1 is found to be 0.3381. The values of the reduced stress resultants at the critical sections are presented in Table 4.2. The variation of the stress resultants with $\frac{y_1}{h}$ are shown graphically in Fig. 4.4.

The values of lower bounds computed from the three solutions for the two shells are presented in Table 4.3 for purposes of comparison. In each solution collapse load p is computed using both the N_c and $M_\phi - N_\phi$ yield criteria. The smaller of these two values is the lower bound of that solution. In like manner the lower bounds are determined for the three solutions and the best lower is indicated in Table 4.3.

Table 4.2 : Reduced stress resultants at critical sections for shell 2 (Medium Shell)

ϕ	$\frac{N_x}{N_o}$ (x=L/2)	$\frac{N_\phi}{N_o}$	$\frac{M_\phi}{M_o}$	$\frac{N_{x\phi}}{S_o}$ (x=0)
0	-0.4205	-0.1642	-1.1992	0.0000
5^o	-0.4836	-0.1624	-1.1496	0.6172
10^o	-0.6460	-0.1566	-0.9892	1.3980
15^o	-0.8287	-0.1451	-0.7404	2.4321
20^o	-0.9033	-0.1265	-0.4218	3.6635
25^o	-0.6966	-0.1001	-0.0905	4.8252
30^o	0.0009	-0.0683	0.1925	5.3809
40^o	0.8500	-0.0200	0.1200	4.1000
45^o	1.4518	-0.0044	0.0000	2.5133

Table 4.3 : Lower bound solutions

Shell type	Lower bound solution 1 (p in kg/m^2)		Lower bound solution 2 (p in kg/m^2)		Lower bound solution 3 (p in kg/m^2)		Best lower bound (p in kg/m^2)
	N_c criterion	M_ϕ-N_ϕ criterion	N_c criterion	M_ϕ-N_ϕ criterion	N_c criterion	M_ϕ-N_ϕ criterion	
Long Shell (Shell 1)	896.0	1859.0	1024.0	2440.0	946.0	2640.0	1024.0
Medium Shell (Shell 2)	2504.0	1044.0	2870.0	1557.0	2085.0	1920.0	1920.0

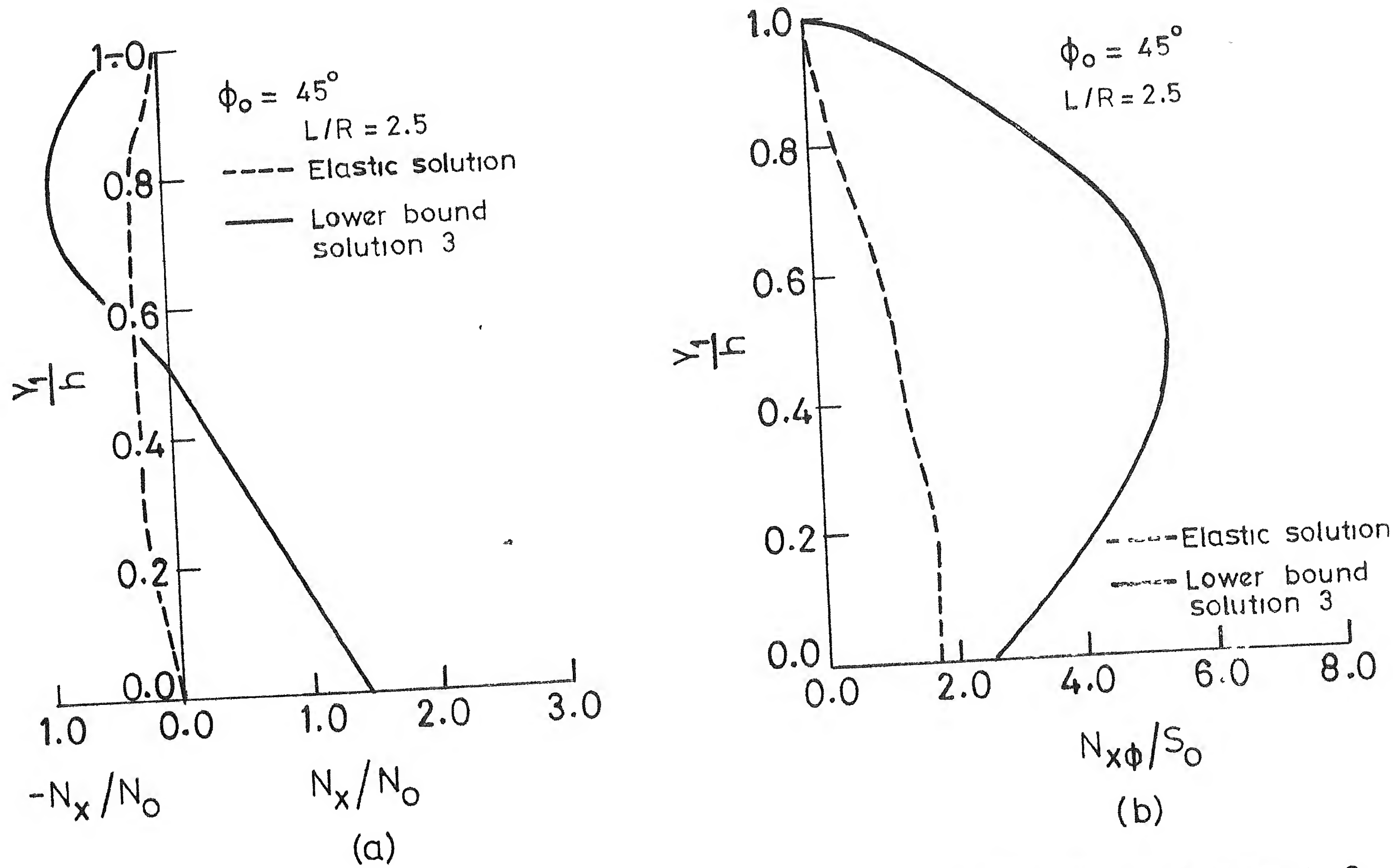
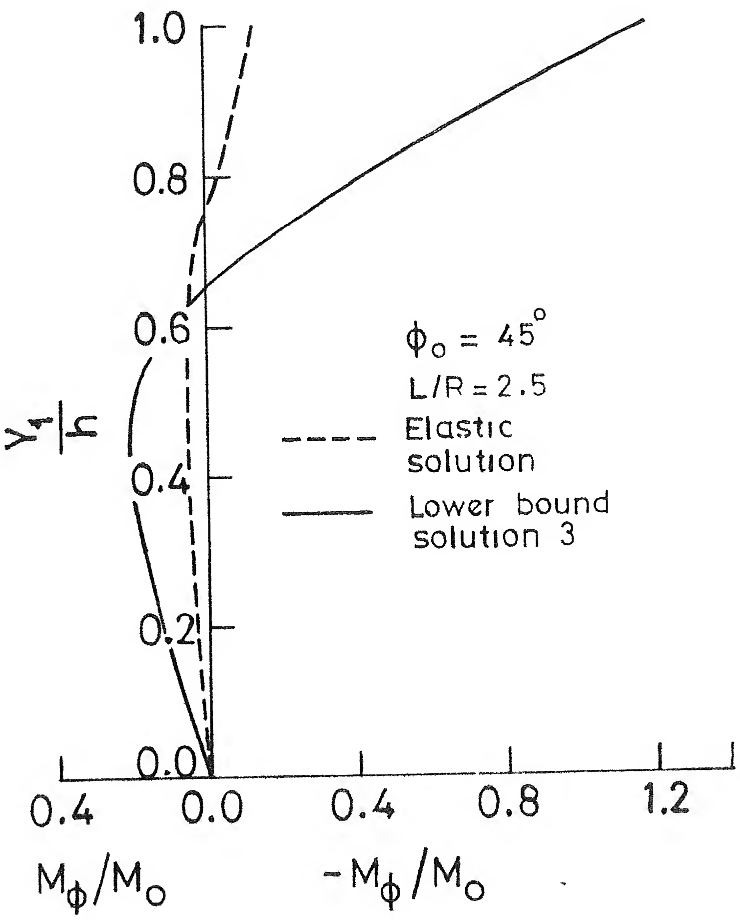
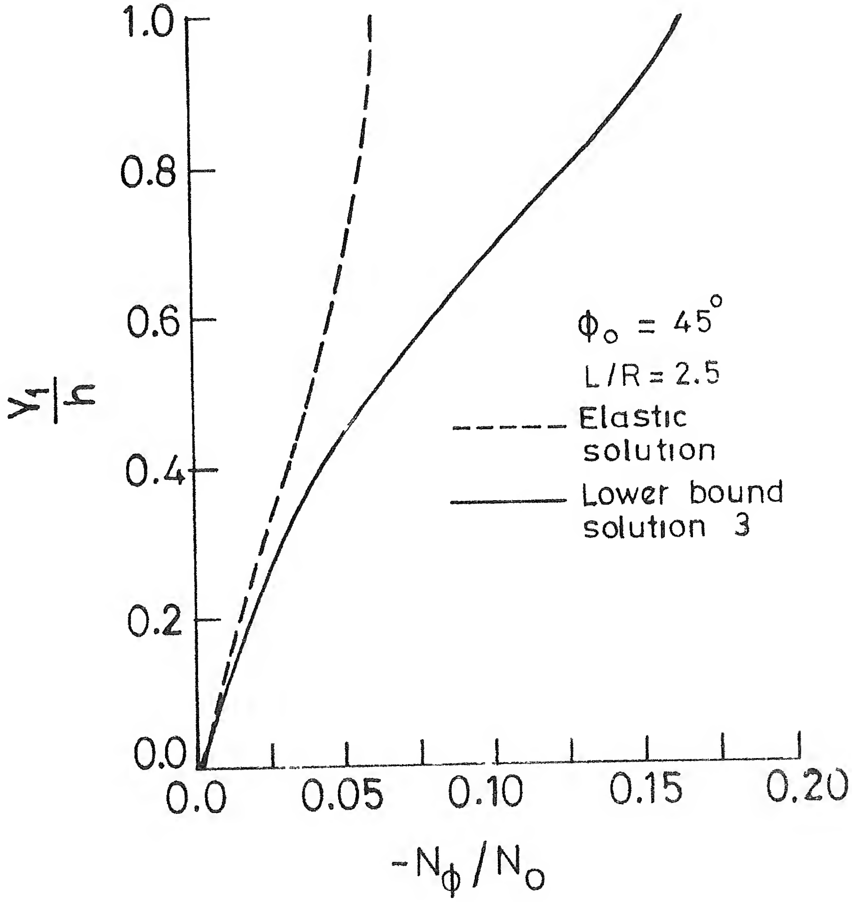


FIG. 4.4 DISTRIBUTION OF STRESS RESULTANTS FOR SHELL 2



(c)



(d)

FIG.4.4 DISTRIBUTION OF STRESS REULTANTS FOR SHELL 2

4.9 Design of Shells

In this section the design procedure for simply supported RC cylindrical shell roofs with longitudinal edge beams is described. Two shells are designed one for each mode of collapse. Elastic design is also carried out for working loads.

The design procedure is indicated as follows :

(a) For a given working load, the design ultimate load will be estimated using suitable load factors. For simplicity of calculations, distribution of live load can be taken as gravity loading.

(b) The amount of longitudinal steel in the edge beam will be preassigned. Elastic design may be taken as a guide. The factor η can be taken as equal to 0.2.

(c) The value of β_1, which gives optimal lower bound will be determined in the lower bound solution 3.

(d) Collapse load will be computed from the three lower bound solutions. Out of these three values the maximum value will be picked up and the lower bound solution that yields the maximum value of the collapse load will be noted.

(e) Depending on the relative values of the design and computed values of the limit load, σ_c and t will be revised.

(f) The stress resultants will be computed using the corresponding lower bound solution.

(g) The transverse steel will be computed from Eq. (2.29) so as to follo' the force distribution at different points on the shell.

(h) Longitudinal and diagonal reinforcement will be computed form the distributions of N_x and $N_{x\phi}$ respectively.

In all these calculations, the stress in the steel will be the yield stress σ_{sy}.

To compare with the elastic design, two shells with different geometric parameters are designed. The parameters of the two shells are :

Shell 1 (Long Shell) : $\phi_o = 30^o$, $\beta = \frac{2}{3}\phi_o$, $\frac{L}{R} = 3.333$ and $R = 8.0$ m. The edge beam dimensions are : $2W = 25.0$ cms and $2H = 150.0$ cms.

Shell 2 (Medium Shell) : $\phi_o = 45^o$, $\beta = \frac{2}{3}\phi_o$, $\frac{L}{R} = 2.5$ and $R = 8.0$ m. The edge beam dimensions are : $2W = 30.0$ cms and $2H = 67.50$ cms.

In the elastic design the two shells are designed for a working load of 400 kg/m^2. Limit design is carried out for an average load factor of 2.5 for both the shells. Shell 1 (Long shell) fails due to the N_c-yield criterion and the solution 2 gives the best lower bound. Shell 2 (Medium shell) fails due to the M_ϕ-N_ϕ yield criterion and the solution 3 gives the best lower bound.

Table 4.4 : Comparison between the elastic and limit designs

	Thickness t (in cms)	σ_c (in kg/cm^2)	Transverse steel (in cm^2/m)	Longitudinal steel (in cm^2)	Diagonal steel (in cm^2/m)	Longitudinal steel in the edge beam (in cm^2)
Shell 1						
a) Elastic design	8.0	150.0	2.50	-	11.50	102.0
b) Limit design	8.0	150.0	5.50	52.0	18.00	75.0
Shell 2						
a) Elastic design	8.0	150.0	3.58	-	9.30	58.0
b) Limit design	9.0	150.0	11.50	14.0	12.0	30.0

For the two shells under consideration, thickness of the shell, compressive strength of the concrete and the quantity of steel at critical sections are presented in Table 4.4. The quantity of the longitudinal steel shown indicates the steel required to resist the total longitudinal tension developed at the centre of the span. The transverse steel is estimated at crown to resist M_ϕ and N_ϕ per unit length. Diagonal steel per unit length is computed to resist the diagonal tension equal to $N_{x\phi}$, developed at the critical section which is at the support.

From the foregoing analysis it can be concluded that the long shells fail due to the N_c-yield criterion and the medium shell fails due to the M_ϕ-N_ϕ yield criterion.

CHAPTER 5

NUMERICAL ANALYSIS OF CYLINDRICAL SHELL ROOFS

5.1 General

In the last two chapters closed form solutions are developed for the limit analysis of cylindrical shell roofs with or without edge beams. In all the solutions, the distribution of one of the stress resultants is preassigned. Hence the solutions are not unique. It is always possible to improve the lower bound by resorting to some other distribution. The success of the method depends mainly on the skill of the analyst in selecting physically appropriate function to represent the stress field. The closer the guess function is to the actual solution, presumably the closer is the lower bound to the actual collapse load.

With this in view a numerical method of analysis is developed. In this method, the continuous stress fields of the structure are replaced by a finite number of parameters by rewriting the basic differential equations of equilibrium in finite difference form. A linear or linearized nonlinear yield condition written in terms of the stress parameters is introduced. It is well known that any problem of plastic stress analysis of a structure can be recast as a problem in linear programming. The load parameter is maximized in

such a way that nowhere in the domain of the shell, the yield condition is violated. This method is widely applied to the analysis of frames and plates. The section that follows gives a brief review of the literature.

5.2 Selective Literature Survey

The variational principles of Greenberg and Prager (27) for the limit analysis of elastic-perfectly plastic structures require the optimization of a linear functional subject to linear constraints. Dorn (28) solved the plastic limit analysis of plane pin-jointed trusses using the linear programming technique. Koopman and Lance (29) first applied linear programming technique to continuous structures like plates with various boundary conditions. Hodge and Belytschko (30) applied non-linear programming technique for the limit analysis of plates. Both lower and upper bounds are found out for plates of various boundary conditions. Grierson and Graham (31) used linear programming, to compute the collapse load of plate structures by kinematic method. Application of linear programming to determine the collapse loads for plane frames is also found in the literature (32).

In the literature almost all the attention is paid to the mathematical programming techniques applied to the frames and plates. From the foregoing it is clear that little attention is paid to the analysis of cylindrical shells by

mathematical programming techniques. The mathematical programming techniques are extensively covered in the literature (33, 34, 35). For completeness the method of linear programming is comprehensively presented in the following section.

5.3 Linear Programming

The methods applied to find optimum value of a function are known as optimization or mathematical programming techniques. These methods have wide application in many areas including the field of engineering. Linear programming is a branch of mathematical programming techniques, where in the objective function and the constraints are linear functions of the variables. The brief literature review presented in the Section 5.2 suggests the application of this method in the context of the limit analysis of structures.

A linear programming problem can be stated as follows : Defining a function Z_o usually known as an objective function as

$$Z_o = \sum_{j=1}^{r} k_j x_j , \qquad \ldots (5.1)$$

it is to be found the value of x_j, $j = 1,\ldots,r$ such that Z_o is maximized or minimized. x_j has to satisfy the constraints of the form ;

$$\sum_{j=1}^{r} a_{ij} x_j \left\{ \begin{matrix} \leq \\ = \\ \geq \end{matrix} \right\} b_i, \quad i = 1,\ldots,m , \qquad \ldots (5.2)$$

where only one sign holds for each constraint. The k_i, a_{ij}, and b_i are assumed to be constants. In addition, the value of the variable x_j should be positive. Thus

$$x_j \geq 0 \qquad \text{for all } j. \qquad \ldots (5.3)$$

In linear programming, constraints mean the restrictions of the form (5.2).

5.3.1 Unrestricted Variables

As already said that the variable x_j should be non-negative. But in most of the linear programming problems some or all of the variables can take any sign. In such situations a variable x_j is expressed as difference of two variables as

$$x_j = x'_j - x''_j ,$$
$$x'_j, x''_j \geq 0. \qquad \ldots (5.4)$$

Depending on the relative values of x'_j and x''_j , x_j can be negative, zero or positive. But in the solution to the linear programming problem either x'_j or x''_j will be present (55) but not both simultaneously. If x'_j is present $x''_j = 0$ and vice versa.

5.4 Formulation of the Problem

5.4.1 Finite Difference Method

Finite difference method is used to replace the differential equations of equilibrium (2.15) through (2.18)

by algebraic equations. In doing so, a second order curve is assumed to find out first and second derivatives of a function. If y_i, y_{i+1} and y_{i-1} represent three consecutive function values at an interval of Δx, then the first and second derivatives at the point i are given by (24)

$$\frac{dy_i}{dx} = \frac{y_{i+1} - y_{i-1}}{2\Delta x} , \qquad \ldots (5.5)$$

$$\frac{d^2y_i}{dx^2} = \frac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} \qquad \ldots (5.6)$$

In case if (i+1) is a point on the boundary, then the first derivative at the point (i+1) is given by

$$\frac{dy_{i+1}}{dx} = \frac{y_{i-1} - 4y_i + 3y_{i+1}}{2\Delta x} \qquad \ldots (5.7)$$

The expressions (5.5) through (5.7) are used for first and second derivatives in this thesis.

5.4.2 Equilibrium Equations

To generalize the method the equilibrium equations are transformed into nondimensional form. With this in view, the following transformations are made to convert the variables into nondimensional quantities :

$$\bar{x} = \frac{x}{L} \text{ , } r = \frac{R}{L} \text{ , } \bar{t} = \frac{R}{t} \text{ , } s_o = \frac{S_o}{N_o} \text{ ,}$$

$$n_x = \frac{N_x}{N_o} \text{ , } n_{x\phi} = \frac{N_{x\phi}}{S_o} \text{ , } m_\phi = \frac{M_\phi}{M_o} \text{ and } n_\phi = \frac{N_\phi}{N_o} \text{ ,} \qquad \ldots (5.8)$$

where $\bar{x}$ is nondimensional coordinate in X direction, r and $\bar{t}$ are nondimensional shell parameters, s_o is the ratio of S_o and N_o, n_x and n_ϕ are nondimensional inplane direct forces, $n_{x\phi}$ is the nondimensional inplane shear force and m_ϕ is nondimensional bending moment. Further, if p_W is the intensity of the working load and λ is a load factor then p_o, which is the collapse load in nondimensional form, is given by

$$p_o = \frac{\lambda\, p_W}{\sigma_c} \text{ .} \qquad \ldots (5.9)$$

Using the foregoing transformations, the equilibrium equations (2.15) through (2.17) are simplified as

$$r \frac{\partial n_x}{\partial \bar{x}} + s_o \frac{\partial n_x}{\partial \phi} = 0 \text{ ,} \qquad \ldots (5.10)$$

$$\frac{\partial n_\phi}{\partial \phi} + s_o\, r \frac{\partial n_{x\phi}}{\partial \bar{x}} - \frac{1}{4\bar{t}} \frac{\partial m_\phi}{\partial \phi} = -\lambda \frac{p_W}{\sigma_c} \bar{t} \sin\phi \text{ ,} \qquad \ldots (5.11)$$

$$n_\phi + \frac{1}{4\bar{t}} \frac{\partial^2 m_\phi}{\partial \phi^2} = -\lambda \frac{p_W}{\sigma_c} \bar{t} \cos\phi \text{ .} \qquad \ldots (5.12)$$

Since Q_ϕ is not a generalized force, it is eliminated by using the equilibrium equation (2.18).

Further, the differential equations of equilibrium (5.10) through (5.12) are replaced by algebraic equations using the finite difference transformations (5.5) and (5.6). Since both the geometry and loading of the shell are symmetric, only a quadrant of the shell is considered for the analysis as shown in Fig. 5.1a. Referring to the Fig. 5.1b, let i and j locate the position of a grid point and let $\Delta\bar{x}$ and $\Delta\phi$ be the mesh sizes in the x and ϕ directions respectively, then Eqs. (5.10) through (5.12) at any grid point (i, j) are expressed as

$$\frac{r}{2\Delta\bar{x}}\left[n_{x(i+1,j)} - n_{x(i-1,j)}\right] + \frac{s_o}{2\Delta\phi}\left[n_{x\phi(i,j+1)} - n_{x(i,j-1)}\right] = 0, \quad \ldots (5.13)$$

$$\frac{1}{2\Delta\phi}\left[n_{\phi(i,j+1)} - n_{\phi(i,j-1)}\right] + \frac{s_o\, r}{2\Delta\bar{x}}\left[n_{x\phi(i+1,j)} - n_{x\phi(i-1,j)}\right]$$

$$- \frac{1}{8\bar{t}\Delta\phi}\left[m_{\phi(i,j+1)} - m_{\phi(i,j-1)}\right] = -\lambda\frac{p_w}{\sigma_c}\bar{t}\sin\phi , \quad \ldots (5.14)$$

$$n_{\phi(i,j)} + \frac{1}{4\bar{t}(\Delta\phi)^2}\left[m_{\phi(i,j+1)} - 2m_{\phi(i,j)} + m_{\phi(i,j-1)}\right]$$

$$= -\lambda\frac{p_w}{\sigma_c}\bar{t}\cos\phi . \quad \ldots (5.15)$$

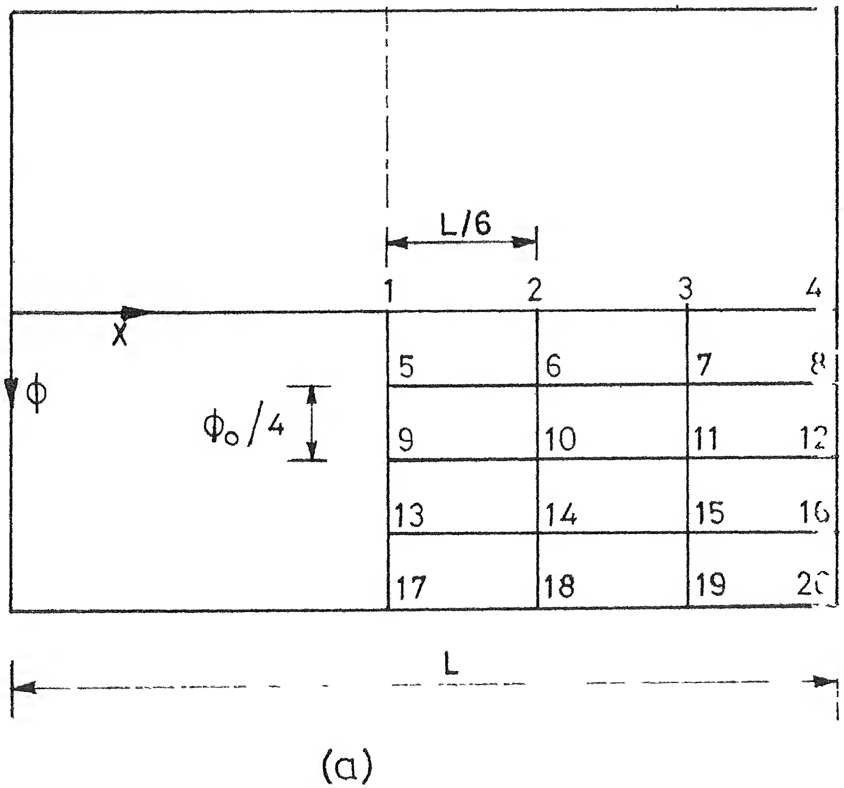


FIG.5.1 PLAN OF A SHELL AND GRID POINTS DIVISION

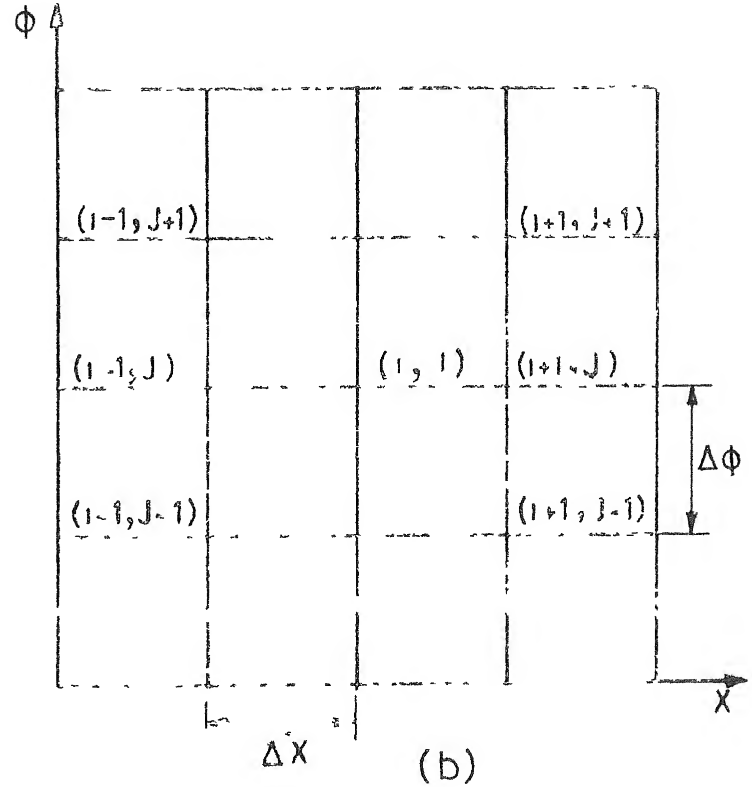


FIG.5.1 FINITE DIFFERENCE GRID

5.4.3 Symmetry and Boundary Conditions

The stress resultants n_x, n_ϕ and m_ϕ are symmetric and $n_{x\phi}$ is antisymmetric. For shells with free longitudinal edges the following boundary conditions are used :

(i) at $x = 0, L$; $n_x = n_\phi = m_\phi = 0$,

... (5.16)

(ii) at $\phi = \phi_0$; $n_{x\phi} = n_\phi = m_\phi = 0$.

5.4.3 Matrix Notation

At each grid point, the type of equations (5.13) through (5.15) have to be generated. As such there will be a number of equations for the part of the shell under consideration. For convenience these equations are expressed in the matrix form. This form is beneficial for computer analysis.

If the unknown stress resultants at the grid points are taken as variables and are represented by $x_1, x_2, x_3, \ldots, x_n$, and if $\{X\}$ define a column vector, then $\{X\}^T$, the transpose of the vector is expressed as

$$\{X\}^T = (x_1, x_2, \ldots, x_n). \qquad \ldots (5.17)$$

Let $[A]$ define a rectangular matrix of size m x n where m is the number of equilibrium equations and n is the number of unknown stress resultants. Let $\{D\}$ define a

$\{X_r\}$ is a vector of size m x 1 containing the variables corresponding to the independent columns,

$\{X_s\}$ is a vector of size (n-m) x 1 containing the variables corresponding to the dependent columns.

Multiplying Eq. (5.19) by $[B]^{-1}$ on both sides

$$[B]^{-1}[B]\{X_r\} + [B]^{-1}[C]\{X_s\} = \lambda\,[B]^{-1}\{D\}, \quad \ldots (5.20)$$

or

$$\{X_r\} = \lambda\,\{E\} - [F]\{X_s\}, \quad \ldots (5.21)$$

where

$$\{E\} = [B]^{-1}\{D\},$$

$$[F] = [B]^{-1}[C]. \quad \ldots (5.22)$$

Thus the variables in the vector $\{X_r\}$ are expressed in terms of the variables in the vector $\{X_s\}$.

5.4.4 Objective Function and Constraints

Any linear programming problem has to optimize a function while satisfying a set of linear inequalities or equalities (the constraints) of the form (5.2) and non-negativity restrictions of the form (5.3). In the present problem, the load factor λ is taken as an objective function.

Thus

$$Z_o = \lambda \ , \qquad \ldots (5.23)$$

in which λ is taken as a variable and its value will be found such that Z_o is optimum.

The constraints are formulated such that the stress resultants or a combination of stress resultants should not exceed a specified limit at each grid point. The stress resultants acting at each grid point are N_x, $N_{x\phi}$, N_ϕ, M_ϕ and Q_ϕ. Here Q_ϕ is not a generalized stress and hence no restriction is necessary on it.

In Chapter 2, the N_c and M_ϕ-N_ϕ yield criteria are proposed. In the N_c-yield criterion, N_x in compression zone is limited to N_c and there is no restriction on N_x in the tension zone. In all the closed form solutions, the magnitude of N_x in the tension zone is limited by specifying the position of the neutral axis and satisfying the over all equilibrium of the shell. In the present numerical analysis if no such restriction is made on N_x, then the solution becomes unbounded. Similarly $N_{x\phi}$, should also be restricted in magnitude. M_ϕ and N_ϕ are restricted by the yield condition given by Eq. (2.30). These conditions give constraints at each grid point.

Therefore the constraints at each grid point are obtained as

$$- N_c \leq N_x \leq \alpha_1 N_c \,,$$

$$- \alpha_2 S_o \leq N_{x\phi} \leq \alpha_2 S_o, \qquad \ldots (5.24)$$

$$- b \leq \frac{M_\phi}{M_o} - c \frac{N_\phi}{N_o} \leq b,$$

where the values of the factors α_1 and α_2 are preassigned from practical consideration.

The constraints (5.24) are made nondimensional by dividing the first constraint by N_o and the second constraint by S_o. Using the relations (5.8) to express the stress resultants in dimensionless form and since $N_c = M_o = \sigma_c t$, the constraints (5.24) are expressed in nondimensional form as

$$- 1 \leq n_x \leq \alpha_1 \,,$$

$$- \alpha_2 \leq n_{x\phi} \leq \alpha_2 \,, \qquad \ldots (5.25)$$

$$-b \leq m_\phi - c\, n_\phi \leq b.$$

Further the constraints are modified to fit into the linear programming problem as

$$n_x \leq \alpha_1 \,,$$

$$n_{x\phi} \leq \alpha_2 \,,$$

$$m_\phi - c\, n_\phi \leq b, \qquad \ldots (5.26)$$

$$- n_x \leq 1 \,,$$

$$- n_{x\phi} \leq \alpha_2,$$

$$- (m_\phi - c\, n_\phi) \leq b.$$

This type of constraints have been developed at each grid point. While developing the constraints, the matrix relation (5.21) is used to express the variables (stress resultants) that are contained in the vector $\{X_r\}$ in terms of the variables contained in the vector $\{X_s\}$. Thus all the constraints are expressed in terms of the variables in the vector $\{X_s\}$. This clearly shows that the number of variables and consequently, the size of the matrix is reduced. If no such elimination is done as given by the matrix relation (5.21), the equilibrium equation given by the matrix relation (5.18) can be treated as equality constraints, which requires additional computer memory. While treating equality constraints, artificial variables are to be added to form the initial basis. Further, since the load factor λ is taken as a variable, when the vector $\{D\}$ is transferred to the left side, the right hand side contains zero vector in the relation (5.18). If the equality constraints contain zeros on the right hand side, artificial variables enter the basis at zero level without improving the objective function which becomes a degenerate problem. This process will be continued till all the artificial variables are eliminated from the basis. Hence convergence becomes poor because of the roundoff errors.

After formulating the constraints, they can be expressed in the matrix form as

$$\left[G \right] \{ X_s \} \leq \{ Y \}, \qquad \ldots (5.27)$$

where the matrix $[C]$ is formed by the coefficients of the constraints and the vector $\{Y\}$ represents the right hand side of the constraints given by inequalities (5.26).

Thus the objective function and the constraints are given by the relations (5.23) and (5.27) respectively for the problem of linear programming.

5.3 Computer Programme and Results

A computer programme has been developed in which the following aspects are dealt with, viz., (i) generation of the elements in the various matrices $[A]$, $[B]$, $[C]$, $[\Gamma]$ and vectors $\{D\}$ and $\{E\}$, (ii) identification of the variables in the vectors $\{X_r\}$ and $\{X_s\}$, (iii) generation of the elements in the matrix $[G]$ and the vector $\{Y\}$.

The revised simplex method (35) is used to solve the linear programming problem. The constraint matrix $[G]$ and the vector $\{Y\}$ in the matrix relation (5.27) are given as a data to the revised simplex method computer programme. The inequalities in the relation (5.27) will be converted into equalities after adding slack variables. The values of the objective function Z_o, the variables in the vector $\{X_s\}$ and the slack variables are given as an output in the computer programme.

The values of the variables in the vector $\{X_r\}$ are determined from the matrix relation (5.21) by back substitution.

Thus the optimal value of the load factor λ and the stress resultants at different grid points are evaluated.

5.5.1 Numerical Example

A cylindrical shell with free longitudinal edges is analysed to illustrate the numerical technique. The geometric parameters of the shell are : $\phi_0 = 45^0$, R = 8.0 m, $\frac{L}{R} = 2.5$ and t = 8.0 cms. In the analysis p_w and σ_c in the equation 5.9 are taken as equal to 300 kg/m^2 and 150 kg/cm^2 respectively. To apply the finite difference scheme, the shell is discretized as a grid with mesh size of $\Delta\bar{x} = \frac{1}{6}$ and $\Delta\phi = \frac{\phi_0}{4}$ as shown in Fig. 5.1a. The factors α_1 and α_2 in the constraints (5.26) are taken as 1.5 and 3.0 respectively in the present problem.

In Section 5.8.2 a shell with the same geometric parameters is analysed. The lower bound solution 3 is the best solution and the load factor is equal to 1.950. The present numerical solution given a load factor of 1.972. The reduced stress resultants that are obtained at different grid points by the numerical solution are given in Table 5.1. The distribution of stress resultants by both the analytical and numerical methods is shown in Fig. 5.2 for purposes of comparison. N_x distribution is drawn for $x = \frac{2}{3}$ L and for

Table 5.1 : Reduces stress resultants at different grid points

Grid Point No.	$\frac{N_x}{N_o}$	$\frac{N_{x\phi}}{S_o}$	$\frac{N_\phi}{N_o}$	$\frac{M_\phi}{M_o}$
1	-0.2545	0.0	-0.3472	-1.6400
2	-0.50	0.0	-0.4303	-1.7444
3	0.0	0.0	-0.3108	-1.7724
4	0.0	0.0	0.0	0.0
5	-0.15	0.0	-0.3350	-1.6296
6	-0.3140	-0.8100	-0.3377	-1.6290
7	0.0	-1.6168	-0.3230	-1.6288
8	0.0	-2.3435	0.0	0.0
9	0.1275	0.0	-0.1293	-0.6900
10	0.2500	-0.9875	-0.1378	-0.7036
11	0.0	-1.9738	-0.1400	-0.7200
12	0.0	-3.0	0.0	0.0
13	0.2120	0.0	-0.0600	-0.2550
14	0.4400	-0.8950	-0.0640	-0.2480
15	0.0	-1.4150	-0.0650	-0.2450
16	0.0	-2.70	0.0	0.0
17	0.4975	0.0	0.0	0.0
18	1.00	0.0	0.0	0.0
19	0.0	0.0	0.0	0.0
20	0.0	0.0	0.0	0.0

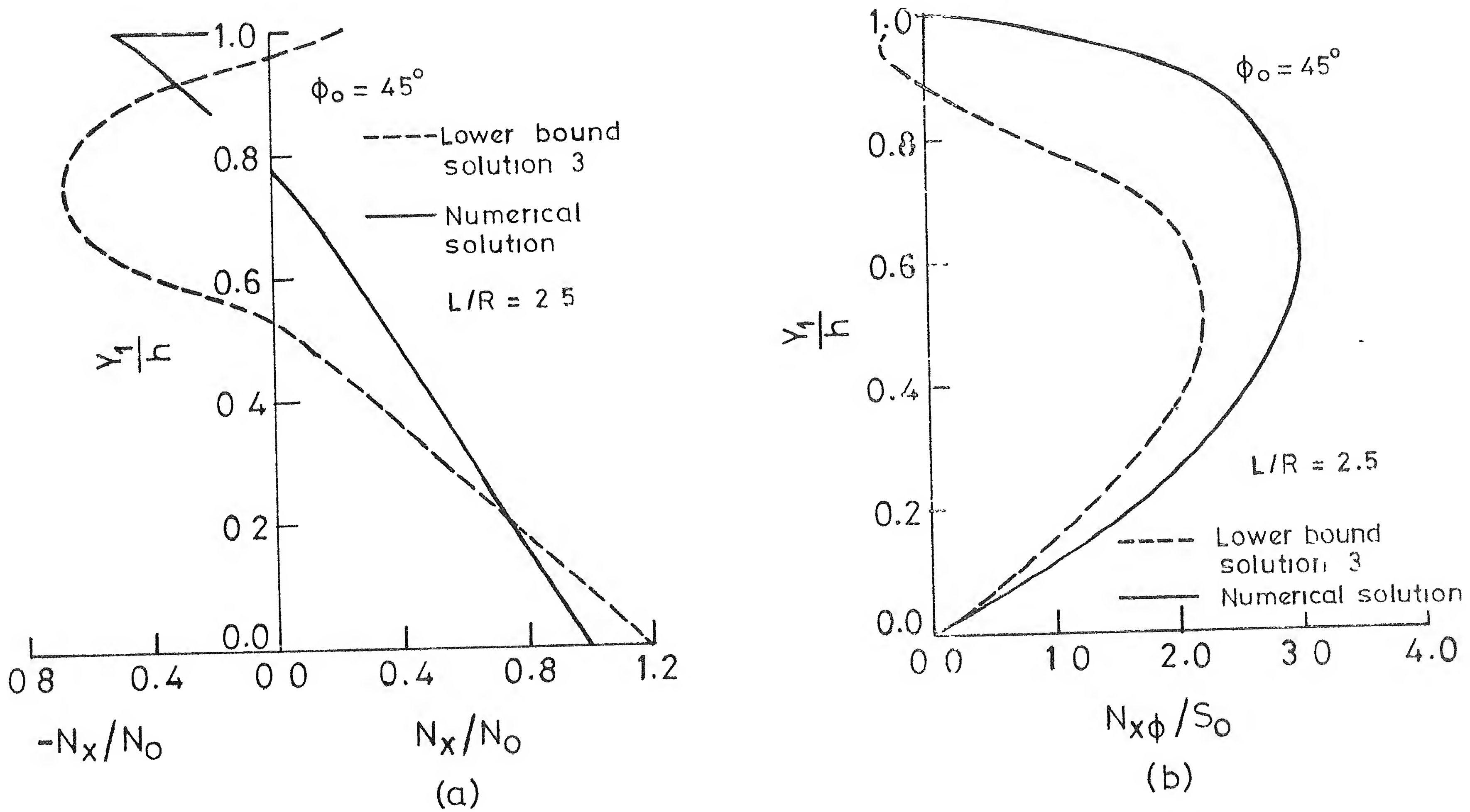


FIG. 5.2 DISTRIBUTION OF STRESS RESULTANTS

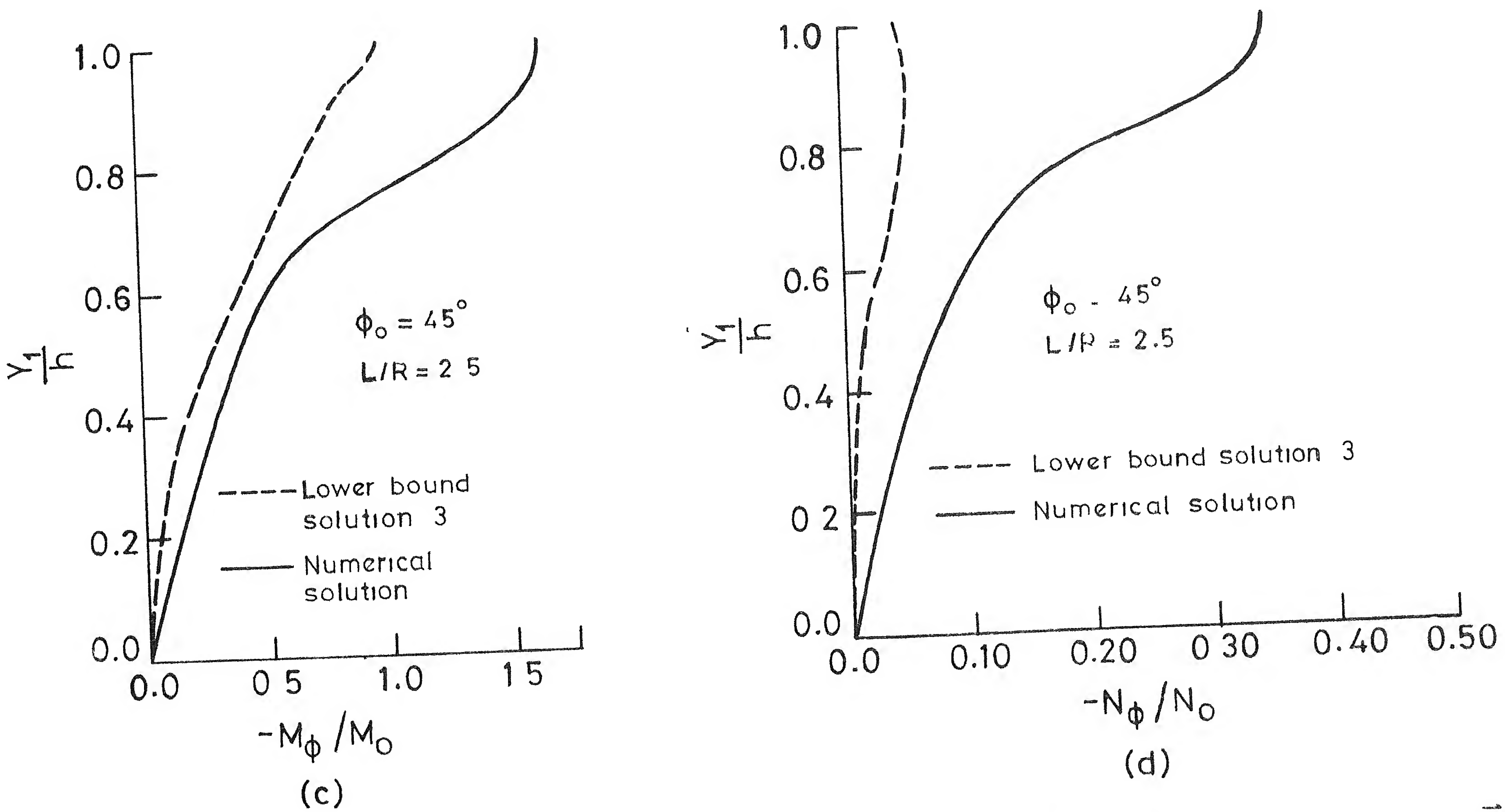


FIG. 5.2 DISTRIBUTION OF STRESS RESULTANTS

$N_{x\phi}$ it is drawn at support. M_ϕ and N_ϕ distributions are drawn for $x = \frac{L}{2}$.

5.6 Discussion of Results

Numerical analysis is carried out for the shell with two different mesh sizes. In the first one $\Delta\bar{x} = \frac{1}{6}$ and $\Delta\phi = \frac{\phi_0}{3}$ and in the second mesh $\Delta\bar{x} = \frac{1}{6}$ and $\Delta\phi = \frac{\phi_0}{4}$ are considered. Values of 1.210 and 1.972 for λ are obtained in the first and second case respectively.

In the present numerical analysis double precision accuracy is necessary in computer programme. It is observed that the single precision value for the load factor is about 1.5 times the value obtained with double precision accuracy. Because of the inadequacy of computer storage, further division of the shell into finer mesh is not carried out to test for the convergence of this technique.

From the values of the stress resultants presented in Table 5.1, it is observed that the longitudinal variation of M_ϕ and N_ϕ at a particular angle ϕ is not considerable. This indicates the same trend as obtained in the closed form solutions wherein it is found that M_ϕ and N_ϕ are independent of x. Further the maximum value of M_ϕ obtained in the present numerical solution is about 1.75 times that of the value obtained by the lower bound solution 3 (Fig. 5.2c).

The distribution of N_x (Fig. 5.2a) is linear in both the compression and tension zones whereas in the lower bound solution 3 it is parabolic in the compression zone. In addition, maximum values for N_x occur at $x = \frac{2}{3}L$ in the present numerical solution instead of at centre of span in solution 3. And also it is found N_x becomes zero at all the grid points lying on the transverse cross section at $x = \frac{5}{6}L$.

Fig. 5.2b indicates the distribution of $N_{x\phi}$ as per the numerical solution and lower bound solution 3.

From Fig. 5.2d it can clearly be found that there is a wide discrepancy in the magnitudes of N_ϕ by both numerical and lower bound solutions. The value of N_ϕ obtained by numerical technique is quite high and it is 7.6 times that of the value obtained by lower bound solution 3 at crown.

From the foregoing discussion it can be concluded that the stress resultant N_ϕ is quite sensitive in the present numerical solution. The solutions are converging as the mesh size decreases.

CHAPTER 6

AN OVERVIEW

6.1 Summary and Discussion

This thesis is mainly concerned with the development of lower bound solutions for RC circular cylindrical shell roofs. For the shells under consideration, the ends are simply supported and the longitudinal edges can be either free or with edge beams. The analysis is carried out for uniform gravity loading.

Limit analysis requires a yield condition which characterizes the failure of a material of the structure. To the best of the author's knowledge no yield condition is available in the literature for reinforced concrete cylindrical shells. Because of this fact, two independent yield criteria are proposed viz., (i) N_c-yield criterion, (ii) M_ϕ-N_ϕ yield criterion. The N_c-yield criterian states that a shell fails if N_x reaches a specified value in compression. The M_ϕ-N_ϕ yield criterion states that a shell fails due to the interaction of the transverse moment M_ϕ and membrane direct force N_ϕ. This yield condition is equivalent to the failure of a reinforced concrete section subjected to a moment M and an axial force N (22). To reduce the complexity of the problem the yield curve of the M_ϕ-N_ϕ yield criterion is suitably

linearized by inscribing a polygon. This modification is on the conservative side. A shell fails if at least one of the two yield conditions is satisfied.

In the N_c-yield criterion the failure of a shell is initiated due to crushing of concrete in compression. This may cause a sudden collapse of a shell. To avoid the sudden collapse, the limiting strength of a shell in compression (N_c) can be reduced by multiplying with some preassigned factor. Because of this reduction of N_c, a reduced collapse load is obtained. But in an actual collapse the shell fails due to the yielding of steel in the tension zone before it actually fails due to crushing of concrete.

In the present lower bound solutions, $N_{x\phi}$ is not restricted in magnitude. Diagonal steel will be provided to resist diagonal tension developed due to $N_{x\phi}$. In some occasions where the computed diagonal steel is heavy, a restriction can be placed on the magnitude of $N_{x\phi}$. Code specifications regarding the spacing of the reinforcing bars will fix the magnitude of $N_{x\phi}$ in such situations.

While drawing the yield curve for M_ϕ-N_ϕ yield criterion, the value for reduced reinforcement ratio $\mu(\frac{A_{st}}{\sigma_c}\frac{\sigma_{sy}}{t})$ is taken as equal to 0.236 in Eq. (2.29). This quantity of steel is the value required for a balanced rectangular section under pure flexure as per I.S. Code (25)

by ultimate strength design. The contants b and c which define the straight line equation given by Eq. (2.30) are computed as 0.833 and 2.25 respectively. For any other value of μ, the yield curve can be drawn using Eq. (2.29) and the corresponding values of b and c in Eq. (2.30) can be determined.

Three lower bound solutions are developed for shells with and without longitudinal edge beams. In all these solutions, the distribution of N_x is preassigned in the transverse direction and the other stress resultants are solved using the differential equations of equilibrium, boundary and continuity conditions.

For shells with edge beams the transverse moment M_ϕ at the junction of the shell and edge beam is taken as zero. And also it is considered that the edge beam cannot resist horizontal forces. These assumptions are the same as in the elastic analysis (37). The distribution of longitudinal force across a cross section of the edge beam is taken as trapezoidal (fully in tension). To determine the collapse load the edge beam reinforcement has been preassigned. Elastic design can be taken as a guide. The factor η specifies the distribution of force in the edge beam (Fig. 4.1c). It is observed that as η decreases, collapse load increases. A minimum value of 0.2 for η is taken in the present study.

In all the lower bound solutions the angle β which defines the position of neutral axis is preassigned. It is found from the analysis of shells as the angle β increases collapse load increases. Therefore the value of the angle β be limited such that the longitudinal steel can be accommodated in the tension zone ($\beta \leq \phi \leq \phi_o$). In the present study β is taken as equal to $\frac{2}{3}\phi_o$.

In the lower bound solution 3 (Sections 3.6 and 4.7) the angle β_1 which gives the position of maximum value of N_x in the compression zone ($0 \leq \phi \leq \beta$) is taken as a variable while other parameters of a shell are treated as constants. For shells with free longitudinal edges the optimal values of β_1 which give the best lower bounds are presented in Table 3.1. For shells with longitudinal edge beams the optimal value of β_1 depends on the factors namely depth of the edge beam and the amount of longitudinal steel in the edge beam. Therefore the optimal value of β_1 has to be determined for each problem.

For shells with free longitudinal edges the variation of nondimensional load p_o with $\frac{R}{L}$ ratio for different values of ϕ_o are presented by way graphs (Fig. 3.2). There graphs are useful for preliminary design of cylindrical shell roofs without edge beams. From the graphs (Fig. 3.2) it can be observed that for $\frac{R}{L}$ ratio greater than some value, shells fail due to the M_ϕ-N_ϕ yield criterion and the collapse load is

constant irrespective of $\frac{R}{L}$ ratio. This limiting $\frac{R}{L}$ ratio beyond which collapse load is constant will have different values for shells with different semicentral angles (ϕ_o).

The distribution of stress resultants for a long shell with free longitudinal edges is presented in Fig. 3.3. This shell has failed due to the N_c-yield criterion. And also it is shown in Fig. 3.4 the distribution of stress resultants for a medium shell which has failed due to the M_ϕ-N_ϕ yield criterion. Similarly the distribution of stress resultants for long and medium shells with longitudinal edge beams is shown in Fig. 4.3 and Fig. 4.4 respectively. Failure of the long shell is governed by the N_c-yield criterion and that of the medium shell is governed by the M_ϕ-N_ϕ yield criterion. Further there graphs give an idea about the distribution of forces at collapse of a shell.

For comparison, two shells with free longitudinal edges (one long and one medium) are designed by both the present limit and elastic analyses. The results are presented in Table 3.5. It can be observed that there is a considerable economy due to reduction of concrete strength, thickness of shell and also quantity of steel as compared to the elastic design. Besides a comparative design statement for shells with longitudinal edge beams is also presented in Table 4.4. For these shells there is no saving of materials but in fact there is an increase of reinforcement as compared to the elastic design.

In Chapter 5 an attempt is made to introduce a numerical technique to obtain lower bound solution for a RC cylindrical shell roof. A comparison is made between the distribution of stress resultants as obtained by both numerical and closed form solutions.

6.2 Conclusions

Two independent yield criteria viz.,(i) N_c-yield criterion, (ii) M_ϕ-N_ϕ yield criterion are proposed for RC cylindrical shells (Section 2.3.6). These approximate yield criteria can be adopted till an accurate yield criterion is developed supported by experimental evidence. However the lower bound solutions that are developed in the Chapters 3 and 4 are valid even for a revised yield criterion.

The lower bounds computed from the proposed limit analysis coincides with the upper bound evaluated for a beam mechanism, taking the shell as a beam of thin curved cross section, if the shell fails due to N_c-yield criterion (See appendix). Hence there solutions give the exact limit load if the shell fails due to the N_c-yield criterion.

From the analysis it is observed that long shells fail due to the N_c-yield criterion. For medium shells with higher values of semicentral angle (ϕ_0) say 45^0 and above the M_ϕ-N_ϕ yield criterion governs the collapse.

Lower bound solution 2 always gives a better value than the solution 1. Lower bound solution 1 may be used for preliminary design for the simple reason that the expressions for the stress resultants are fairly simple as compared to the other two solutions.

In general, it can be said from the Fig. 3.2, solution 2 gives the best lower bound if a shell fails due to the N_c-yield criterion. Further, solution 3 gives the best lower bound if a shell fails due to the M_ϕ-N_ϕ yield criterion.

The stress resultants M_x, Q_x and $M_{x\phi}$ are neglected in the present study. This assumption is satisfactory if the span L exceeds its chord width B by 1.0 or over (24). Hence these solutions are satisfactory for shells satisfying the above limitation.

The numerical technique developed in Chapter 5 serves as an alternative to the analysis of a RC cylindrical shell roof at collapse. It avoids the difficulty of selecting various statically admissible stress fields for complicated structures like shell roofs. The stress resultant N_ϕ is very sensitive in the present numerical solution.

6.3 Further Scope

It is proposed :

(1) to develop an accurate yield criterion for RC cylindrical shell with experimental evidence,

(2) to develop a better lower bound solution for cylindrical shell roofs,

(3) to analyse short span cylindrical shell roofs,

(4) to analyse multibarrel shell roofs and

(5) to compare the numerical results by the finite element approach.

APPENDIX

Upper Bound Solution

For completeness, the upper bound solution for simply supported RC circular cylindrical shell roof available in the literature (22) is presented here.

Let N'_p denote the force/unit length in tension zone of a shell. The shell is considered as a beam of curved cross section. The load per unit length P of the shell under a uniform gravity loading of intensity p is given as

$$P = 2pR\,\phi_o\,. \qquad \ldots (1)$$

Assuming a simple beam mechanism with a central plastic hinge, the external work W_E is

$$W_E = \frac{PL\Delta}{2}$$

$$= pRL\,\phi_o\,\Delta, \qquad \ldots (2)$$

where Δ is an arbitrary deflection of the shell at its centre during collapse.

The plastic moment capacity M_p, of the shell as a beam with uniform compression at collapse is

$$M_p = 2N'_p\,R(\phi_o - \beta)\,a, \qquad \ldots (3)$$

where a is the lever arm (vertical distance between the centroids of the compression and tension) and is given as

$$a = \frac{R(\phi_o \sin \beta - \beta \sin \phi_o)}{\beta(\phi_o - \beta)} \; . \qquad \ldots (4)$$

The internal work W_I, during the plastic collapse is

$$W_I = \frac{4M_p \Delta}{L} \; . \qquad \ldots (5)$$

Using the relations (3), (4) and (5)

$$W_I = \frac{8N_p' R^2 \Delta (\phi_o \sin \beta - \beta \sin \phi_o)}{L\beta} \; . \qquad \ldots (6)$$

Equating the external work done W_E and the internal work W_I it is obtained

$$p_u = \frac{8N_p' R(\phi_o \sin \beta - \beta \sin \phi_o)}{L^2 \beta \phi_o} \; , \qquad \ldots (7)$$

where p_u is the upper bound.

From the static equilibrium condition of the shell (algebraic sum of the forces in the longitudinal direction is zero) it is found

$$N_p' = \frac{\beta N_c}{(\phi_o - \beta)} \; , \qquad \ldots (8)$$

where $N_c = \sigma_c t$.

Substitution of Eq. (8) in Eq. (7) yields

$$p_u = \frac{8N_c R (\phi_o \sin \beta - \beta \sin \phi_o)}{\phi_o(\phi_o - \beta) L^2} \qquad \ldots (9)$$

which is same as the lower bound given by Eq. (3.42).

Similarly it can be shown for shells with longitudinal edge beam also. Further it can be shown that the lower and upper bounds are same for other lower bound solutions also when the shell fails due to the N_c-yield criterion.

From the foregoing it is concluded that an exact value for the collapse load is found when the N_c-yield criterion governs the collapse of a shell.

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